Elena S. WENTZEL

# OPERATIONS RESEARCH

## A Methodological Approach

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Е. С. Вентцель

# ИССЛЕДОВАНИЕ ОПЕРАЦИЙ

**задачи**
**принципы**
**методология**

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E. S. Wentzel

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## A Methodological Approach

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*На английском языке*

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# FOREWORD

The aim of this book is to present in a widely appealing form the subject matter and methods of operations research (OR), a science acquiring in the latest years ever wider applicational arenas. It is of a relatively recent origin and so far its boundaries and contents can hardly be deemed rigorously defined.

As a taught discipline, it is included in many curricula though what is covered is not always the same. In professional quarters, it lacks unanimity in much the same way. Some OR professionals treat it merely as the mathematical methods of optimization such as linear, nonlinear and dynamic programming. Others on the contrary, do not include these techniques in OR, approaching it generally from the viewpoint of game theory and statistical decision making. A third group tends to deny that OR exists as an independent discipline preferring to include it with cybernetics—a term inappropriately defined and therefore variously understood. There are even those who tend to endow OR with an enormously wide sense claiming it to be a key science.

Time, obviously, will tell which direction this relatively young discipline will take, which of the methods generally accepted as its constituents will hold, and which will branch off into self-contained dis-

ciplines. A particular question that remains concerns the future relation between OR and systems theory, a widely discussed topic of recent times.

Beyond doubt, however, is the point that whatever the area of activity may be—production planning and inventory control, transportation, military operations and weapon system development, personnel management, social services, health services, communication systems, computer networks and information systems, to name but a few—the problems they pose with ever increasing rate are similarly formulated, can be identified by several features they have in common and, last but not least, can be solved by similar methods. These problems are therefore conveniently grouped under the common heading of OR problems.

A typical situation giving rise to an OR problem is as follows. An undertaking (a system of actions) having a clearly defined objective may be organized in several alternative ways by making a decision based upon choosing from a set of possible alternatives. Each choice offers its own advantages and disadvantages, so that in a complex situation the decision maker might not be able to make a preferable option at once and quickly decide why he should prefer one alternative and not another. To clarify the situation and compare the alternatives in several aspects, OR suggests a series of mathematical operations. Their aim is to analyze the situation critically and thus prepare a decision for those bearing the responsibility for a final choice.

Without being confined to an exclusive area of practice, the discussion in the book will be focused on the methodological aspects which are common to all OR problems wherever they might appear. Therefore the main emphasis will be placed on such methodological treatises as problem formulation, model development,

and assessment of computational results, rather than on mathematical rigor. The experience collected in the field suggests that it is these evaluation-and-analysis steps (not manipulatory) skill] in transformations and computations) that might present major difficulties to an inexperienced practitioner when he will try to implement the mathematical methods of decision making to practical problems of value.

In writing this text the author employed her many-year experience in applying OR to various practical arenas. The standpoint, into which the experience would have inevitably crystallized, was attempted to be delivered in the simplest possible form.

The relevant mathematics is simple and requires some expertise in probability theory. Whenever the text goes beyond this simplicity frame, the needed explanations are immediately provided. As to the crest points of main ideas and methodology, they are treated with the highest rigor, calling for the attention and intense thinking of the reader.

The author tended to avoid as possible an entertaining presentation since an inavoidable curiosity dressing would believingly distract the reader from, rather than help him to, the point. It does not imply, however, that the author would stick to stiffly formal, vapid fashion—a notorious attribute of mathematical texts. Rather, the adopted spirit is fairly jovial and even slipping occasionally to formulations which are not perfectly correct and, hence, open to criticism. The excuse is that the book is devoted primarily to the practitioners who are novices to the subject, rather than to experts. Ample reservations, should they be made to perfect the rigor, would only repell the reader, making it hard for him to grasp the essence of the matter.

The chapters of the book are not equal in difficulty of comprehension and the amount of mathematics involved. The reader who would like to get a general acquaintance with the subject, problems and possibilities of OR may confine himself to attentive reading through Chapters 1 and 2 and initial sections of the rest chapters. Beyond these introductory chapters, an intelligent reader will find what to calculate, thus having a closer look at the methods of OR and making himself familiar with the relevant mathematics to feel at ease if met with it in specialized issues.

Chapter 6 dealing with queueing theory considers a series of methodologies almost entirely absent from nonspecial literature; that is the reason for its comparatively large size.

The aim of the book is to provide the reader with a foundation upon which to build a further study of a particular topic. References which cover extended applications and more formal mathematical treatment of the subject matter are provided at the end of the book.

*E.S. Wentzel*

# Chapter 1

# THE NATURE AND USE OF OPERATIONS RESEARCH

## 1.1 Operations Research—What It Is and What It Does

Now that the revolutionary influence of science and technology is felt almost everywhere, ever more weight is put by science on planning, management and control. There are many reasons for this. The modern technology grows more sophisticated, scales of ventures and their aftermath effects spread dramatically, automatic control being implemented in many fields, all this calls for the analysis of complicated, objective-bound processes to be performed from the viewpoint of their structure and organization. The science is required to deliver recommendations of how such processes should be controlled in optimal (judicious) fashion. Far behind are those times when a correct and efficient control had been found by trial-and-error approach. Now that the losses due to possible errors may cost too much to be afforded, the control must be developed scientifically.

Requirements of practice called for special scientific methods which can conveniently be grouped under a common heading, operations research (OR). Referring to this term we shall imply the application of quantitative, mathematical methods to prepare decisions bound to be made in all the fields of objective-bound human activity.

To get a more closer look at what we mean under decision, consider a venture headed towards a certain target. The person (or a group of persons) who has initiated the venture always has a set of alternative courses of action, say, to pick up tools and build up a process in some way or another, to allocate resources according to one or another pattern, etc. The *decision* is the alternative that the manager chooses as his course of action. Decisions, consequently, may be bad or good, thought over or hasty, well-founded or arbitrary.

The need for decision making is as old as the hills. Even at times of wilderness a leader of a tribe going to hunt a mammoth has to make a decision on where to place a trap and how to place the hunters and what must their weapon be, etc. We also would make decisions in our everyday life without even noticing it. Leaving, for example, for the office in the morning, we would make a series of decisions: what to wear? should an umbrella be carried? where to cross the street more conveniently? which urban transport to pick? and so on. A manager responsible for a process is to arrive at a decision every time when he has alternatives, say, in placement of manpower, assignment of a work order, to name only a few.

Does this decision making mean that we are involved in operations research? Not at all. Operations research begins whenever one or another mathematical technique is applied to substantiate the decisions being taken. Obviously, in simple situations decisions are taken without any mathematics invoked, simply by the sound judgement and expertise. To decide what to wear going out or where to cross the street mathematics is hardly needed and undoubtedly will not be in the future. An optimum decision is arrived at as if by

oneself, guided by the practical experience. If at times a decision made is not the very best, then what? Mistakes are committed to improve on them.

The decisions we will be concerned with more in this book, however, are those heavily loaded with responsibility. Let, for example, the public transportation network be planned in a new town having its own layout of concentration points such as factories, apartment blocks, etc. Obviously, certain decisions are to be made concerning how to trace the routes and what vehicles should use them, where to place the stops, and what should be the interval in the ordinary and rush hours, etc.

These decisions are much harder to make and, what is more important, many things depend on them. Erroneously chosen they spoil life and have an adverse impact on business activity of a whole town. Certainly, in this situation a decision may well be prepared intuitively from experience and common sense (as is not infrequently done). Yet the decisions are more judicious if backed up by mathematical reasoning. The preparative computations may help to avoid long and costly search of a decision by trial and error.

To expand on this, let us take another example. Let us imagine that we are entrusted with a large scale venture, say, to divert waters from northwardbound rivers to water-hungry arid areas of the south. Should we take an arbitrary, willful decision that might bring about adverse effects on a global scale, or better to make preparative computations with mathematical models? The dilemma could hardly be viewed as ambiguous—the multilevel computations are mandatory.

'Score twice before you cut once' a proverb says. Operations research is the very mathematical scoring

of future decisions, helping to spare time, efforts, and resources, and to avoid blunders, which are no longer such as to improve anything—the cost of respective correction will be too expensive.

The more complex, expensive and large in scale the designed system is, the less allowable in it are willful decisions and the more gain in importance scientific methods which, when implemented, provide an estimate of each decision's consequences, help discard the unallowable versions and recommend the most successful ones. They help in assessing whether the available information is adequate to prepare a correct decision and, if not, then indicate what data should be additionally collected. It would be extremely risky to be guided solely by intuition, i.e. experience and common sense. Modern science and technology evolve so fast that the experience may simply not have been acquired. To say more, the undertakings initiated are often unique and original. Thus, the experience may be virtually absent, and common sense, if not proven by calculation, will be deceptive. The calculations that make the process of decision making easier are the subject matter of operations research.

We have mentioned already that OR is a relatively young science (the notion of "youth", though, is relative in scientific quarters: many scientific disciplines have been nipped in the bud shortly after their appearance, failing to find an application). The name operational research[1] was first used in the years just before the outbreak of World War II to describe an approach

[1] The name 'operational research' is still preferred by British scientists to 'operations research' which was coined in the United States—*Translator's note.*

taken by interdisciplinary groups of British scientists summoned to solve strategic and tactical problems for military management. The decisions that were prepared by the groups were primarily concerned with the application of weapons and distribution of forces and facilities among various targets. Similar problems, though with a differently named approach, had been taken elsewhere, in particular, in the Soviet Union. After the war, this approach spread into a variety of practical fields: industry, agriculture, construction, marketing, social services, transportation, communications, health services, pollution control, etc. There hardly exists an area of human enterprise where mathematical models and OR techniques have not been implemented in some form or the other.

To get a better insight into the specifics of the science, consider several typical OR problems. Intentionally taken to represent various fields, they, although simplified in the setting, bring forth fairly well the main idea of OR scope and objectives.

**1. Transportation problem.** A network of factories consumes certain types of raw materials produced at several sources. The sources are linked with the factories each in its own way, by railway, waterway, motor road, or air, having its own cost per unit of transported load. It is required to design a supply layout, specifying which source should supply what raw material and in what amount, and such that the demand is satisfied at minimum transportation cost.

**2. Construction project planning.** A section of motor road is being built. The construction involves a certain amount of manpower, construction equipment, repair shops of a known capacity, trucks, and so on. It is required to plan the project, i.e. to schedule the order

of works, distribute the equipment and manpower along the construction site, and plan the maintenance and repair works, so that the project is terminated in the shortest term.

**3. Seasonal sales movements.** The entrepreneur of a marketing firm plans to establish a network of branch off stores to market a certain stock of consumer goods. He is to decide on the number of retailing stores, their location, distribution of inventories and personnel at each of the stores so that the business turns out maximum economic efficiency.

**4. Snow fencing of roads.** In the Nordic countries snow storms present serious interference to traffic. Any road blocked by snow means losses for the economy. For road fencing, there exist a few alternatives—to construct the road with a suitable profile, to set up fencing facilities beside the road, etc.—each with its own cost for construction and maintenance. The decision maker has at hand the data on prevailing wind directions and frequency and intensity of snowfalls. He is to decide which means of snow control will be best in cost-efficiency terms, i.e. what means of fencing should be assigned to each of the roads subject to the losses from snow blizzards.

**5. Search for enemy submarines.** An antisubmarine warfare officer receives a message that an enemy submarine has appeared in an area that Coastal Command surveillance. A group of aircraft takes off to find and bomb the submarine. The raid is to be set up in a most rational fashion, that is the routes are traced and altitudes and a pattern of attack are chosen such as to complete the task with the greatest certainty.

**6. Sample tests in quality control.** To guarantee the quality of produced items at a specified level a factory establishes a sampling test system. A sound

system requires that the testing procedure, that is choosing of sample size, sequence of tests, sorting rules, etc., should be devised so as to provide for the quality at a minimum cost.

7. **Localizing an epidemic outbreak.** Reportedly, a communicable desease broke out in an area. It is decided to conduct a medical examination of the area population so as to reveal those infected and infective. The search is to be performed by a certain medical staff equipped with proper facilities. It is required to schedule the search, that is to decide on the number of stations, their positioning, sequence of examination, types of analyses taken, etc., so as to maximize the revealed percentage of infective agents.

8. **Library shelf organization.** A large public library has on its shelves the books in increased demand, in medium demand, and those retrieved very seldom. The books can be rearranged on the shelves and between store rooms and some demands can be readdressed to other libraries. The problem is to work out a retrieval system such as to satisfy the demand in the best possible fashion.

The examples could be piled up further on, yet even the above suffice to reveal the common features of OR problems. Although taken from various applications, the common traits can be readily identified: each time we have spoken of an undertaking striving for a certain objective. Each time certain conditions have been specified to describe the environment, say, the facilities placed at the disposal. And each time within the framework of the posed limitations we are to make a decision such that to bring the undertaking to, in a sense, a most profitable condition.

Accordingly, solution of such problems has been developed on a general basis of the techniques and

mathematical instrumentation which taken together make up the methodology and tooling of OR.

The observant reader might have noticed that not all of the above examples would require mathematical treatment when practically handled. For some of them, a "good" decision taken on a common sense basis would suffice. The trend, however, is to increase the part of problems which require mathematical treatment for a decision to be made. An especially large momentum the mathematical techniques gain where automatic control systems are introduced. A control system, if employed to guide a process, rather than merely reduce data and retrieve information, will be unsound if the process under control has not been mathematically modeled and studied prior to system implementation. The mathematical techniques to prepare decisions come to carry even more weight as scales and complexity of enterprises increase. Air traffic of a small airport, for example, can be easily handled by one expertised traffic controller; a large airport requires an automatic traffic control system which operates according to a set in algorithm. Preparation of such an algorithm always involves some computation analysis performed beforehand, i.e. operations research.

## 1.2 Main Concepts of and Approaches to Operations Research

This section is devoted to the terminology, main concepts, and principles of operations research in its scientific background.

The term 'operation' will imply any undertaking or system of actions which is conducted under a common scheme and headed toward a certain objective. Thus all the undertakings presented in examples 1

through 8 of the previous section may be treated as operations.

An operation is always a controlled action, which is meant to say that we are to decide in what way to select the parameters setting up its structural organization. The organization is to be understood here in a wide sense and includes the technical means involved in the operation.

Any option, that is alternative set, of parameters which in their selection depend upon the will of the manager is referred to as a *decision*. Decisions can be successful and unsuccessful, meaningful and unsound. The *optimal decisions* are those which are preferred to the others for one reason or another. The aim of operations research is to quantitatively prepare optimal *decisions* by finding optimal *solutions* for OR problems.

At times, though it is a relatively seldom occasion, the study yields a sole, strictly optimal solution; still more often, it bounds an area of optimal solutions virtually equal in value, within which a final option is to be made.

Note that the decision making proper does not fall within the operations research domain. It is a manager or, what is more often, a group of persons who is entrusted with the final decision making and who bears the responsibility for the choice. In arriving at a choice, the decision makers may take into account other reasons (quantitative or qualitative in nature) not considered in preparing mathematical recommendations.

Even with totally automatic systems of control which seem to make decisions with no human interference the judgement of a human is present in the form of the algorithm employed by the system. It

should be remembed that the development of the algorithm or selection of one of its plausible versions is a decision making process too, indeed laden with responsibility. With the progress in automatic controllers, the functions of the human are not taken up by a machine, rather, they shift from a basic level to a more intelligent one. To add more weight to the argument, some automatic control systems are developed so that the human may actively interact to aid the process of control.

The parameters whose collection forms the set of a decision are referred to as *controllable* or *decision variables.* These variables may materialize as various numbers, vectors, functions, physical characteristics, etc. To illustrate, if a transportation schedule is worked out for similar commodities from origins $A_1, A_2, \ldots, A_m$ to destinations $B_1, B_2, \ldots, B_n$, then the decision variables are the numbers $x_{ij}$, indicating the amount of commodities from the $i$th origin, $A_i$, to the $j$th destination, $B_j$. A set of numbers $x_{11}$, $x_{12}, \ldots, x_{1n}, \ldots, x_{m1}, x_{m2}, \ldots, x_{mn}$ forms a decision.

Simple OR problems operate with a comparatively short list of decision variables. In a majority of problems of practical value, though, the number of decision variables may be very large, the fact in which the reader is invited to convince himself by trying to evaluate and label the decision variables in examples 1 through 8 of Section 1.1. To make the discussion more simple we shall denote the set of decision variables by a single letter $x$ and spell it 'decision $x$'.

In addition to decision variables which we may control within certain limits, any OR problem contains specified, constraining conditions which must not be violated. Examples of such constraints may be capacity

of a truck, planned production target, weighting characteristics of equipment, etc. They in particular include the means (monetary costs, labour, equipment) that are put under the managerial disposal, or any other conditions limiting the decision. Taken as a whole, they establish what we would call a *set of possible decisions.*

We denote this set again by a single letter $X$, and write down the fact that $x$ belongs to the set symbolically as $x \in X$, read as element $x$ belongs to set $X$.

Restating our goal in mathematical terms, we are to evaluate on a set of possible decisions $X$ those of them $x$ (a single one, or more frequently a region of decisions) which are in one or several aspects more efficient, i.e. more useful or preferable to, than the others. To judge the merits of particular decisions and compare them, a quantitative criterion is introduced, called the *measure of effectiveness* and in practical problems can also be referred to as the *objective function.* This performance index should be chosen such that it describes the objective of the operation under concern. A decision is deemed best when it performs uppermostly in arriving at the objective.

In deriving a measure of effectiveness, $W$, we first of all ought to ask ourselves what we are striving at, embarking on the enterprise. Choosing a decision, we naturally would prefer one that maximizes the effectiveness function, or keeps it at a minimum. Obviously, the profit of a venture should be made a maximum; when the measure of effectiveness is built around costs, it should be minimized. Mathematically we shall describe it as $W \Rightarrow \max$ if the function is to be maximized, and as $W \Rightarrow \min$ if it is to be minimized.

In those situations where the outcome of an operation comes under the influence of random factors—these

might be weather, variations in demand and supply, equipment failures, and so forth—the measure of effectiveness is derived as the mean value (mathematical expectation) of the function which is going to be minimized (maximized). (In more detail such situations shall be discussed in Section 2.2.)

When random factors influence the operations whose objective, $A$, can be either made or not made at all and none of the outcomes in between is of interest, the measure of effectiveness is chosen in terms of the probability to make this goal, $P(A)$. Shooting a target, for instance, on an indispensable condition to demolish it, the measure of effectiveness will be the probability of demolishing.

Setting the index of performance in a wrong way can entail adverse consequences. Operations conducted with an improperly chosen measure in view might incur unjustified costs and losses.

To illustrate how the measure of effectiveness should be derived we come back again to examples 1 through 8 of Section 1.1. Choosing a natural measure, we indicate whether it should be minimized or maximized. It should be kept in mind, however, that not in all of the examples a measure of effectiveness is uniquely brought to the surface by the wording of the problem. Consequently, the reader may deviate in his judging the situations from the viewpoint given below.

**1. Transportation problem.** The problem is to supply raw materials at minimum transportation costs. The measure of effectiveness, $R$, is equal to the total transportation costs per unit time, say, per month; $R \Rightarrow \Rightarrow \min$.

**2. Construction project planning.** The construction is to be planned so that it is completed in a shortest

possible period. It would be natural to choose the period of construction as a measure of effectiveness if it were not for random impacts (equipment failures, delays in completion of certain works, and so forth). The measure, therefore, may be the mean expected time, $\overline{T}$, of completion of the project; $\overline{T} \Rightarrow \min$.

**3. Seasonal sales movements.** For a measure of effectiveness, one may take the mean expected profit $\overline{G}$, gained from the sales over the season; $\overline{G} \Rightarrow \max$[2].

**4. Snow fencing of roads.** The problem is to devise the most economical snow fencing project. Hence, the measure of effectiveness may be chosen as an average for a period (say, for year) of combined costs, $\overline{R}$, for maintenance and repair of the roads, including in that index the basic costs for erection of fences, snow removal costs, and costs due to delayed traffic. Obviously, $\overline{R} \Rightarrow \min$.

**5. Search for enemy submarines.** The objective of the raid is perfectly certain—to sink the submarine; therefore, a measure of effectiveness should be the probability that the submarine will be eliminated.

**6. Sample tests in quality control.** A natural measure of effectiveness brought to the surface by the problem formulation would be the mean testing costs, $\overline{R}$, per unit time, provided that the system of tests guarantees a specified quality level, say, that the mean rejection percentage is not above a specified target value; $\overline{R} \Rightarrow \min$.

**7. Localizing an epidemic outbreak.** An adopted measure of effectiveness may be the average percentage (fraction) $Q$ of the infected and infective people that have been evaluated; $\overline{Q} \Rightarrow \max$.

[2] Henceforth we shall denote a mean value by an overhead bar.

**8. Library shelf organization.** An intently made slip in the formulating of the problem brings some spirit of alternative in finding the appropriate measure of effectiveness. The invitation to satisfy the demand in the best possible fashion does not lead to a straightforward measure. Judging the quality of the service by the time spent in waiting for a book, the measure may be the mean time $\overline{T}$ of waiting, that is, $\overline{T} \Rightarrow \min$. Taking another tack, one may choose as a measure of effectiveness the mean number of books, $\overline{M}$, delivered per unit time; $\overline{M} \Rightarrow \max$.

The considered examples were purposefully selected to be simple enough to make the choice of a measure of effectiveness comparatively straightforward and following obviously from the problem formulation because of its, almost in all cases, unique objective orientation. In practice, however, it is not always the case. The reader would readily find this out by trying, for instance, to choose a measure of effectiveness for an urban transportation system. What should it be: an average speed of moving passengers through the town? or an average number of handled passengers? or perhaps an average mileage a person has to walk because the transport fails to bring him to a wanted place? The problem suggests a lot to think over.

Unfortunately, the choice of a measure for the majority of applicational problems is not an easy task and no ready-made approach can be recommended. It is typical for a problem of the sort that offers some difficulties that a sole number cannot serve as an all-inclusive measure, so other numbers have to be sought to properly describe the effectiveness. We shall look more closely at such multiple criterion problems in Section 2.3.

## 1.3 Model Development

Preparing to accommodate quantitative methods of study, any field requires a mathematical model. To develop a model, a real phenomenon (an operation in our case) is inevitably simplified into a scheme which then is described invoking a suitable mathematical technique. The simple yet still sufficient is the developed model, i.e. the better it describes the phenomenon under study, the more successful will be the study and the more useful the resulted recommendations.

No general approach exists to formulating mathematical models. For any particular situation, a model is developed according to the type of the operation under concern, its objective, and the task of the study (i.e. reflecting the parameters which are to be defined and the impacts to be included). The accuracy and elaboration of a model should be validated against (i) the accuracy required from the solution, and (ii) the data that is at hand or can be additionally collected. If the input data of the computation does not possess a required accuracy, then obviously it would be senseless to go into details, construct an elaborate model and waste time (of both the analyst and a computer) to achieve a precisely optimized solution. Unfortunately, this principle is often overlooked and the models employed are superfluously elaborated.

The mathematical model should involve all the important facets of the phenomenon being studied and all the critical factors on which mainly hinges the success of the operation. On the other hand, a model should be possibly simple and not stuffed with a lot of minor factors whose account complicates the analysis and makes the results hard to grasp. Two traps always

await a mathematician on his way to a model; first, he might flounder in details, or, second, he might adopt too coarse an approximation leaving the essence of the action beyond the model. The trade of model development is an art indeed with the expertise being acquired only gradually as it would in arts.

Since a mathematical model often does not immediately follow from the problem statement, it would be a good practice not to blindly believe in the infallibility of a single model, rather, it would be wise to compare the results yielded by a variety of models so as they may compete. To be sure in a solution, a problem should be attacked several times, from various angles, i.e. with various allowances, technology, and models. If the analytical inference changes from one model to the next only slightly, it is a strong argument in favor of the study having been conducted objectively. On the contrary, if the inferences of different models deviate drastically, the conceptual framefork of the models should be revised, looking upon which of the models describes the action more adequately, and then, if needed, run a test experiment. Typical for an OR analysis is also that a recursive tack is employed to correct the model after a computational run before the next.

Model development is the most important and critical stage of any study; it requires deep insight and understanding into the phenomena being modeled rather than maturity in relevant mathematics. As a rule, "pure" mathematicians who have not managed to consult experts in modeled system trade are able to handle model development only poorly. They would place the main emphasis on the relevant mathematical tooling rather than on the practical side of the problem.

The collected experience has made evident that the most successful models have been developed by either the specialists in the field who have been at ease with the appropriate mathematical techniques or the teams recruited of practitioners in the field and mathematicians. Practitioners in many fields, engineers, biologists, medical experts to name only a few, find it very useful to consult a mathematical expert in OR when they meet with a necessity to make decisions on a scientific basis. These consultations are advantageous to both sides as the mathematicians get acquainted with practical problems from diverse trades. To tackle them mathematicians often have to boost their expertise and extend, generalize and modify the known techniques.

The mathematical education of a practitioner who would like to perform an independent OR study in its own field should involve a variety of mathematical disciplines. In addition to a classical college course of mathematical analysis, OR often employs modern and relatively recently evolved techniques such as linear, nonlinear, and dynamic programming, game theory, decision making, queueing theory, and some others. This book is supposed to supply the reader with a certain grasp in these techniques.

An especial attention among the educational disciplines should be devoted to probability theory: easy operating with statistics and probability patterns is to be preferred to an academic comprehension. The stress on this discipline can be explained by the fact that the majority of operations has to be conducted under the conditions of incomplete certainty, when the performance and outcome depend upon random factors. Unfortunately, it is a rare occasion in engineering, biological, or medical quarters that a specialist

has a firm and easy comprehension of probability. The concepts and rules of the theory find only formal application with no due perception of their sense and spirit. It is infrequent that the theory is viewed to be a sort of magic wand yielding information from nothing, i.e. from total ignorance. Those who think so are under a misapprehension since probability theory is used but to transform data on observed phenomena to infer the behaviour of those which cannot be observed. The reader is supposed to possess the basic knowledge in the theory.

With the above list of mathematical disciplines involved we have not the smallest intention to scare the reader off. First, all things are difficult before they are easy, and any instrument can be mastered if the motivation is strong enough. Second, not all the techniques need be applied to a problem at once; they can be learned one by one starting with that which is needed most. An answer to what problems call for what knowledge can be again obtained from this book.

In developing a model, the complexity of mathematical tools may be different depending on a type of operation, objectives of the study and accuracy of input data. In the simplest situations algebraic equations can suffice. If they are more involved, say, calling for a phenomenon to be considered in its dynamics, then differential equation (ordinary or partial) technique is implemented. In most intricate cases, when both running an operation and its outcome depend on a large number of intimately interrelated random factors, the analytical techniques fail altogether and the analyst has to employ Monte Carlo methods of statistical modeling (discussed in Chapter 7). Loosely speaking, the idea of this approach may be described

as follows. A computer simulates the process of an operation development with all the random variables involved. This manipulation of the process yields an observation (realization) of one random operation run. One such realization gives no grounds for decision making, but, once a manifold of them is collected after several runs, it may be handled statistically (whence the term statistical simulation) to find the process means and make inferences about the real system and how, in the mean, it is influenced by initial conditions and controllable variables.

Both analytical and statistical models are widely implemented in operations research. Each of the models possesses its own advantages and disadvantages. The analytical models are more rough, account for lesser number of variables and are always in need for some allowances and simplifications. On the advantage side, though, they yield more meaningful results which better reflect the relevant regularities. Above all, however, they are better amenable to search of optimal solutions.

The statistical models are more accurate as compared with the analytical, they are also more elaborated, do not require those robbing assumptions, and can account for a large (theoretically unlimited) number of factors. Yet, these methods possess their own disadvantages, they are bulky, poorly analyzable, need a lot of computational time, and, above all, are very difficult at optimal solution search which is performed by trial and error.

The novices in the field whose expertise in OR studies is comparatively small often start a study with, not indispensable, constructing a statistical model, making effort to incorporate as many factors as available. They forget a simple thing—getting a model and running

the computations is but a half way to a useful result; more important is to succeed in analyzing the computational printouts so that they might be translated to the rank of recommendations.

Best works in the OR field were based on the joint application of analytical and statistical models. The analytical model helps to generally perceive a phenomenon, and outlay, as it were, a contour of major regularities. Any further improvements can be achieved with statistical models (in more detail, see the topic in Chapter 7).

To conclude, we are bound to say a few words about the so-called simulation theory. It is applied to the processes whose performance is to be directed from time to time by intervening human will. The manager, or management, in charge of an operation is to make decisions depending on the situation being reported similar to that how a chess player analyzing the game chooses a move. The decision is then computed over by the mathematical model to evaluate the system's response to this decision in a bit of time. The next decision will be made already with a due account of a new real state of the system, and so on. As a result of the manifold repetition of the procedure the manager as if gains the expertise, and by improving on his own and others' mistakes comes gradually to make correct decisions, even if not exactly optimal, then somewhere close to that. These procedures having come to be known as operational gaming became recently very popular and widely recognized as a valuable tool in operational training of managerial staff.

# Chapter 2

# APPROACHING OPERATIONS RESEARCH PROBLEMS

## 2.1 Two Ways of Posing the Problem. Deterministic Models

An operations research problem can be posed in either of two ways. We shall call them, accordingly, primal and inverse. A *primal problem* answers the question: what will be if under specified conditions we make a decision $x \in X$? In particular, what will amount with this $x$ the value of a chosen measure of effectiveness, $W$, or a set of such measures?

To solve such a problem the researcher develops a mathematical model which represents one or several measures of effectiveness as functions of specified conditions and controllable variables.

An *inverse problem* answers another question: how to find such a solution $x$ that the measure of effectiveness, $W$, be a maximum?[1]

Basically, primal problems are simpler to solve than inverse. The latter obviously cannot be solved, without first solving the primal. Some type of operations evolve an easily solved primal problem so that it requires no special attention. In other operations, the model development and measure of effectiveness computation are fairly nontrivial, as, for example, is

[1] In the following we shall always discuss the maximum search situation whatever it may be. Obviously, the minimum case becomes that of maximum by simply changing the sign of $W$, i.e., $\min W = \max(-W)$.

the case with queueing theory we shall discuss in Chapter 6.

A few more details on the inverse problem are in order. If the number of feasible solutions constituting the set $X$ is small, then we may simply evaluate the value of $W$ for each of them to compare and pick up that one (or those) which maximize $W$. Hence, an optimal solution is found by exhaustive search. When the number of feasible solutions constituting the set of possible decisions $X$ is large, however, a blind search by exhaustion is cumbersome, and often virtually impossible. To deal with such situations, the researcher has to invoke direct search methods. These find an optimal solution by successful trial approximations with each next outcome appearing closer to the sought optimum than the previous. We shall discuss some of these techniques in Chapters 3 and 4.

At the moment we confine ourselves to formulating the problem of finding the optimum solution in its most general form.

Assume that we are conducting an operation, whose success we can influence by making a decision $x$ in one way or another (as will be recalled, $x$ is a group of parameters rather than a number). Let the effectiveness of the operation be defined by a single index, $W \Rightarrow \max$.

Assume further that we deal with the simplest, so-called deterministic, situation when all the conditions of the operation are known beforehand, i.e. no uncertainty is involved. Then all the variables on which the performance of the operation depends may be broken down into two groups:

(i) those specified beforehand and thus known (conditions on operating the action); we shall shortly denote them by the Greek letter $\alpha$;

(ii) those under our control, constituting collectively a solution $x$.

Note that the first group of variables contains, among others, the constraints (occasionally referred to as restrictions) imposed on a solution, i.e. defines a region of feasible solutions $X$.

The measure of effectiveness $W$ depends on variables of both groups, or putting it as a formula

$$W = W(\alpha, x) \tag{2.1-1}$$

Considering Eq. (2.1-1) it should not go unnoticed that both $x$ and, generally, $\alpha$, are collections of values (i.e. they are vectors), functions or similar, rather than numbers. Commonly present among the specified conditions $\alpha$, the constraints imposed on controllable variables assume the form of equations or inequalities.

Assuming that the form of Eq. (2.1-1) is known, that is the primal problem has been solved, we may formulate the inverse problem as follows.

Given a set of conditions $\alpha$, find a solution $x = x^*$ such that maximizes the measure of effectiveness $W$.

We denote this maximum by

$$W^* = \max_{x \in X} \{W(\alpha, x)\} \tag{2.1-2}$$

which is read as: $W^*$ is the maximum value of $W(\alpha, x)$ taken over all solutions belonging to the set of all possible solutions, $X$.

We should accustom ourselves to the equations of the type. They will be of help in the following (in Chapter 4, for instance).

Now, what we have got is a typical problem on searching a maximum of a function or a functional.[2]

[2] It should be recalled that the functional is a mathematical quantity which depends upon a function. When a solution $x$ includes functions as well as numbers, then $W(\alpha, x)$ is a functional.

It belongs to the well developed class of variational problems of mathematics. The easiest of them (maximum and minimum problems) are familiar to any graduated engineer. Seeking a maximum, or minimum (an extremum, to be concise), of a function of many variables we commonly differentiate it with respect to all the variables (controllable variables in the discussed circumstances), set the derivatives to zero, and solve the resulted set of equations. Seemingly, nothing could be more simple. Yet, in operations research this classical approach works poorly for several reasons. First, when the variables are many, solving the relevant set of equations appears often a more complex task than the direct search for an extremum. Second, with some constraints imposed on controllable variables, an extremum more often than not lies at the boundary of the region $X$ rather than at the point of zero derivatives. This brings about specific difficulties of the multidimensional variational problem under constraints, whose complexity might occasionally render it not amenable to a modern computer. Besides, in some problems the function $W$ has no derivatives at all—it is specified, say, only for integer-valued arguments. All the mentioned circumstances make the search of extremum not so simple a procedure as it might seem at first glance.

The strategy of the search for extremum and the related optimal solution $x^*$ should always be devised paying due regard to the form of both the function $W$ and imposed constraints. To exemplify, if $W$ is a linear function of controllable variables, $x_1, x_2, \ldots, x_n$, and the constraints imposed on the variables have the form of linear equations or inequalities, then we arrive at a *linear programming* (*LP*) *problem* which is solved by comparatively simple and, what is more important,

standard procedures (see Chapter 3). If $W$ is convex, special techniques of *convex programming* are employed. In problems where the only nonlinearities are quadratic, *quadratic programming* is invoked (see [8]). To optimize the control of multistage activities, *dynamic programming* techniques are implemented (see Chapter 4). Finally, there exists a family of numerical methods devoted to the search of extrema; they typically use algorithms and are therefore well suited for implementation on a digital computer. Some of them involve a random search procedure which not infrequently appears more efficient than that of ordered exhaustion for the case of multidimensional problems.

Summarizing, we can say that the search for optimal solution in the simplest, deterministic environment is a purely mathematical problem of the calculus of variations variety (both constrained and unconstrained), which may offer computational rather than principal difficulties. The procedure is not that straightforward, however, when the problem involves uncertainty.

## 2.2 Decision Making under Uncertainty

In the previous section we considered the optimization problem of OR for the deterministic case, i.e. for the measure of effectiveness depending on two groups of factors, those specified and known beforehand $\alpha$, and controllable variables, $x$. More often than not real problems involve another group, that of unknown factors which in their generality we will denote by a single symbol $\xi$. Now, the measure of effectiveness $W$ depends on all the three groups, or symbolically

$$W = W(\alpha, x, \xi) \tag{2.2-1}$$

Since $W$ depends on unknown factors $\xi$, it cannot be computed explicitly even though $\alpha$ and $x$ are given. Hence, the search for optimal solution is no longer certain. We cannot after all maximize an unknown quantity. Still, we cannot help making this quantity possibly great. Some people do succeed in the circumstances when the entire situation is not obvious, don't they? Occasionally they do indeed. Converting this into mathematical language, we pose the following problem.

> Given the conditions which must be satisfied, $\alpha$, and allowing for the unknown factors $\xi$, find a solution $x \in X$ such that the measure of effectiveness, $W$, be possibly large.

This formulation can no longer be termed rigorous, indeed it contains a proviso "possibly". The presence of uncertain factors translates the problem onto a new level where decisions are to be made under uncertainty.

Let us have a closer look at the new problem. First of all, we ought to confess that uncertainty carries with it no advantage. If the conditions of operations are not known we can no longer optimize the decision with that degree of success we could if adequate information were at hand. Therefore any decision made under uncertainty is worse than that made in the conditions known beforehand. Anyway, we are bound to make a decision, whether it be good or bad. Our task then is to supply the decision with as much judgement as possible. Thomas L. Saaty, an outstanding expert in OR, has his reason putting a bit of irony in his definition of the trade as the art of giving poor answers to practical questions which other methods answer still worse [1].

We are confronted with the necessity of making decisions under uncertainty at every step. For example,

bound for vacation trip we are packing a suitcase. Its dimensions along with the weight of the packing and the set from which we are to choose the things are given (conditions $\alpha$), and the weather at the destination place is not known beforehand (conditions $\xi$). Which clothes ($x$) should we take along? This problem, although similar outwardly to those of OR, will be handled, certainly, with no recourse to mathematics, though we shall invoke some statistical data, say, on the most probable weather in the area and our own tendency to, e.g. catching cold. Whether consciously or subconsciously, we somehow optimize the decision. Interestingly, people seem to employ for that different measures of effectiveness. A young person would rather tend to maximize the sum of expected pleasures (let us put aside how this can be estimated in a quantitative fashion), whereas an elderly traveller would more likely minimize the probability of catching cold.

Consider now a more serious problem. A selection of goods is planned for sales. The salesman would like to maximize the profit. However, he cannot tell in advance how many customers will be shopping and what each of them will need. What will he do? There is no escaping uncertainty, and yet a decision has to be made!

Another example. A dam is planned to be built to protect a region being flooded. Neither moments of floods, nor their water levels are known beforehand. Yet the project must be planned and no uncertainty can relieve us of the necessity to do it.

Finally, an even more complex problem. A weaponry development is planned over a several years horizon. Neither a particular opponent, nor his weapon at the time is known. A decision, however, must be made.

So that to take such decisions judiciously rather than at random, by rush of inspiration, modern science suggests several methods. Which of them to adopt depends on what is the nature of unknown factors $\xi$, what is their origin, and who controls them. Putting it another way, with what type of uncertainty we meet in a particular study?

The reader can reasonably ask if it is really possible to impose type qualification on uncertainties. As turns out, it may be done.

First of all consider the uncertainty of the, so to say, most favorable type for the study. This turns up when the unknown factors belong to the domain of probability theory, that is, are random variables, or functions, whose statistical characteristics are known or can in principle be obtained by an appointed time. We shall refer to these OR problems and relevant uncertainty as *stochastic*.

We should better demonstrate this type of problems by means of an example. Let us imagine that we reorganize the work of a canteen with the intention to enhance its throughput. No exact data are given about how many customers will visit it over working hours, when they will appear, what meals they will order, and how long each of them will be served. However, the required characteristics, if not available so far, can be obtained by statistical inference.

Another illustration. It deals with the design and operation of a vehicle repair and overhaul facility. The objective is to cut down idle time due to break down in service and repair. The facility is to handle the unanticipated repair, hence, the failures, time in repair or overhaul are all factors of random nature. However, the characteristics of all random variables involved can be gained from the respective statistical data.

We get a better sense of this "favorable" uncertainty if we consider it in still more detail. Let the unknown factors be random variables whose distribution densities, mathematical expectations, variances, or other statistical characteristics are principally known.[3] Then the measure of effectiveness, a function of these characteristics, will itself be random. A random variable, however, cannot be maximized since it remains random at any option $x$, that is it cannot be put under control. What course will we choose?

A surface idea that comes to mind first is to substitute mean values (expectations) for the respective random factors $\xi$. The problem then becomes deterministic and can be solved by ordinary techniques.

To say least, the approach is fairly tempting and in some situations even justified. Indeed, the majority of applicational problems in physics, mechanics, and engineering is at every step tackled along these lines, neglecting the random nature of some parameters involved (heat capacity, inductance, friction coefficient, etc.) in that their mean values are employed. The point is essentially how much random these parameters are; if they deviate of their means only slightly, we may use the means and be justified to do so. It is like this in operations research: we may neglect randomness in some problems. For instance, if we are engaged in planning raw material supplies for several production facilities (see Example 1 in Section 1.1), we may, to a first approximation, neglect, say, that real output of supply sources may randomly fluctuate (provided that the bulk supply is well organized). The same obviation of random variables by their

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[3] If among the variables of $\xi$ are stochastic functions, they can be converted to series of random variables.

expectations will be inappropriate, however, if the randomness can substantially impact the outcome of operation. To illustrate, we take a most primitive example of shooting a target. If after making several shots resulted in randomly placed hits we took instead their mathematical expectation, i.e. the target itself, we would be guaranteed to hit the target with each shot which is obviously wrong.

Another example with a lesser evidence. It deals with the design and operation of a vehicle repair facility that is to service, e.g., a government motor pool or a taxi cab company. Should we not plan the randomness of breakdowns (i.e. change the random time of failure-free operation for its mathematical expectation) and the randomness of repair time, then the capacity of the shop would simply not be up to the demand (the full evidence of that will be presented in Chapter 6). Recurrent, though, are operations where randomness is basically involved so that there is no driving them to a deterministic end.

Consider now a situation in which the factors $\xi$ are "substantially" random and therefore markedly influence the measure of effectiveness which, hence, is substantially random, too.

An intuitively sound approach is to operate with the mean of the random measure, i.e. take $\overline{W} = E[W]$ and choose a decision $x$ such that the averaged measure be a maximum, viz.,

$$\overline{W} = E[W(\alpha, x, \xi)] \Rightarrow \max \qquad (2.2\text{-}2)$$

Note that this is the very approach we have adopted in Section 1.2 when we took mean profit and mean time rather than simply profit and time as a measure of effectiveness for problems with uncertainty. For the majority of situations, this approach (we should

call it 'optimization in the mean') is perfectly justified. Really, there is more judgement in a solution chosen so as to maximize the measure of effectiveness in the mean than in one taken at random.

We may ask after all how far have we succeeded in obviating the uncertainty? Certainly, to some extent it is still present. The effectiveness of each operation performed with particular values of random factors $\xi$ can markedly deviate from that expected, assuming both larger and, unfortunately, lesser values. We may only content with the thought that the payoff after multiple repetitions and the optimization in the mean will be larger than that yielded with no computations at all.

This optimization in the mean is often used in practical stochastic OR studies with no investigation into its validity. To be valid, however, the operation to which it is applied must be of a recurrent variety so that a loss in the measure of effectiveness incurred in one situation might be compensated by its surplus in another. To illustrate, if we perform a long series of similar operations with the objective to maximize the profit, then profits of the operations are to sum up so that a minus in one operation can be met by a plus from another with the positive net result.

Yet those who think that this condition will always guarantee the safe side are far from being true. To prove, consider an example of automatic control system planned to manipulate the ambulance car fleet in a big city. The calls from various districts arrive at random at the central control board to be retransmitted to particular, local, ambulance stations from where cars will be sent. It is required to develop a dispatching algorithm such that will enable the ambulance services to operate with the best efficiency.

To this end we need first of all to devise a measure of effectiveness, $W$.

Obviously, the time a patient waits for the doctor is desired to be possibly short. However, this time is a random variable. Should we apply the optimization in the mean, then we need to choose the algorithm that minimizes the mean expectation time. If we adopt this approach, some of our patients would run into real trouble, since the expectation times cannot be added—too long waiting of one patient cannot be compensated for by a quick serving of another. Hence, if we adopted mean waiting time $\overline{T}$ as a measure of effectiveness, we would tend to prefer the algorithm which, although watching for short mean waiting time, might agree with very long waiting times for some of the callers. To avoid these unagreeable circumstances we can augment the measure of effectiveness by a requirement calling for the real expectation time $T$ be not above some largest value $t_0$. Since $T$ varies at random we may not boldly require that the condition $T \leqslant t_0$ be satisfied; we may only require that it be satisfied with such a high probability as to make the event $T \leqslant t_0$ virtually true. Proceeding in this way, we can adopt the value for the probability, $\beta$, which is close to unity, say 0.99 or 0.995, rendering the event practically true. Now we can require that the condition $T \leqslant t_0$ be met with a probability not lesser than $\beta$, or

$$P\,(T \leqslant t_0) \geqslant \beta \qquad (2.2\text{-}3)$$

This constraint implies that the region of feasible solutions $X$ excludes those of them which fail to satisfy the condition. The constraints of the type (2.2-3) will be referred to as stochastic. If present, they substantially complicate the optimization.

The researcher should be particularly cautioned in applying the optimization in the mean to a solitary, or even unique, operation rather than recurrent. The applicability in such situations depends on what results from the failure of the operation, i.e. a low value of effectiveness index brought about; occasionally it may imply a catastrophe. What reason to hope for large payoff in the mean if this particular action might turn out bankruptcy. Again, these catastrophic results may be avoided by introducing stochastic constraints. At a substantially large confidence coefficient $\beta$ we may be practically guaranteed of not being ruined.[4]

Now that we have shortly considered the case of "favourable" (stochastic) uncertainty and generally elucidated the relevant optimization we are still halfway to a complete management of uncertainty—there are worse situations to come. Stochastic uncertainty makes the situation almost certain if the random variables involved are supplied with the respective probabilities. The situation is more evasive when the unknown factors cannot be statistically described. This may happen either when (i) a probability distribution for $\xi$ exists basically, but cannot be evaluated by the moment of decision making, or when (ii) no distribution for $\xi$ exists at all.

An example of situation (i) may be a layout of a computer information system dedicated to serve randomly arrived demands. The probability characteristics of these arrivals would have been inferred statistically if this system or similar had been in operation

[4] The method of probability theory, as applied to practice, is essentially based on the assumption that low probability events are impossible (this topic and the principles underlying the assignment of significance levels $\alpha = 1 - \beta$ are discussed in [2]).

for a sufficiently long time. However, by the scheduled term of design no such information is available. How then will we manage the project decision making?

Certainly, we can, upon some thought, intuitively assign some characteristics to the random factors $\xi$ and optimize a solution $x$ in this background, i.e. as above, in the mean, or by imposing stochastic constraints. This treatment would, undoubtedly, be better, though not very much, than those with a decision made at random. A more sound approach would be to set some decision variables $x$ free to vary, then option a solution in full awareness of its being not the best one and implement it. As the implementation expertise piles up, we could consistently change the free decision variables striving to increase the effectiveness. The algorithms of this type, improving in the process of control, are labeled *adaptive*. Their advantage is in that they unburden us from the need to collect prior statistics, and are liable to readapt in response to changes in the environment. With more expertise, such an algorithm gradually improves similar to how the human improves on committed mistakes.[5]

Now we turn to the most difficult and unpleasant situation (ii), that of no probability characteristics for $\xi$, which, thus, cannot be deemed random in the ordinary sense. A little comment is here in order on the difference between uncertainty and randomness. As will be recalled, probability theory refers the term 'random event' to the events which recur and, more important, have a property of *statistical stability*. The latter implies that similar trials with random outcome tend to assume a stable distribution upon mani-

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[5] For a mathematically rigorous discussion on adaptive algorithms, we refer a mathematically mature reader to [31].

fold repetition. The frequencies of the events tend then to the respective probabilities, and the sample means to the mathematical expectations. Tossing a coin many times, say, the frequency with which heads appear gradually stabilizes and ceases to be random. In weighing a body many times with a balance, the average result ceases swinging and becomes more uniform. These are examples of stochastic uncertainty, "favorable" in our terms.

There is, however, nonstochastic uncertainty which we will loosely label "unfavorable" in reference to its analytical "amenability". This time the factors $\xi$ are intangibles. They are as ever unknown beforehand, but additionally there is no point in speaking of, or trying to evaluate, their distributions or other probability characteristics.

Let us illuminate it by an example. Assume that we plan a production-and-sales operation which depends in its success on what will be the most popular length of ladies skirts, $\xi$, in two years. A probability distribution for $\xi$ cannot be inferred principally of any statistical data. Even if we turn back to history and start accounting for the multiplicity of experiments (years) in which $\xi$ experienced variations, it would hardly help in our outlook. A probability distribution of $\xi$ simply does not exist, by much the same reason as does not exist a set of homogeneous experiments on which the quantity could have a required stability. Hence, we face the situation of "unfavorable" uncertainty.

However untractable mathematically this situation might seem, it would be unwise to give up all mathematical efforts, as even in these adverse circumstances a preliminary computation might be worthwhile.

Let us trace this statement in more detail. Any

unfavorable uncertainty compels us to content with guesswork about $\xi$ only. If we make a right use of it, i.e. choose more or less plausible values for $\xi$, then the problem under study casts into deterministic framework and may be handled by ordinary techniques—indeed attractive and easy seem deterministic situations now!

Still, we have not enough reasons for joy. Assume that having consumed much effort and time (both our own and computer's) we have arrived at some plausible $\xi$. Would then a decision made for these $\xi$ be equally good for other $\xi$? As a rule, it would not. Therefore its value is highly limited, and a good practice at this stage will be to choose a tradeoff decision rather than optimal for some conditions. This decision though not optimal for, possibly, any conditions will be nevertheless applicable throughout their entire range.

Whatever feasible theory of tradeoff does not exist so far, although some efforts in this direction are made in game theory and decision making (see Chapter 8). A common practice is to leave the option of a compromise decision for the OR analyst. He would run the problem many times for various $\xi$ and $x$ to estimate with these preliminary computations the strong and weak points of each version and make an option. The procedure does not necessitate (though it is interesting at times) the knowledge of conditional optima for each set of conditions $\xi$. Calculus of variation techniques step back in this case.

Another valuable aspect of preparatory analyses in the studies involving unfavorable uncertainty should be emphasized. At this preliminary stage the analyses help screen off the decisions $x \in X$ that at any conditions $\xi$ are inferior to those retained and, hence, uncompetitive. It may substantially reduce

the set $X$, at times to a few alternatives amenable to consideration and assessment by the decision maker in search of a suitable tradeoff.

In remedying the unfavorable uncertainty it is always worthwhile to stage competition of various approaches and viewpoints. Among the latter one singles out owing to its mathematical stand—that of utmost pessimism. It states essentially that the decision maker working under unfavorable uncertainty must always expect the worst possible outcome and make the decision yielding best effect in the worst conditions. If in these conditions we gain a payoff $W = \overline{W}$, then we may guarantee that at any other conditions it will not be less (the principle of guaranteed result). This approach is attractive in that it explicitly formulates the optimization problem and thus paves the way for correct mathematical techniques. Yet it is justified far not always. Its major application area covers so-called conflict situations where the conditions are decided by an intelligent person (opponent) who responds to each our decision in the worst way for us.[6] In more "neutral" situations the principle of guaranteed gain is not the only possible and can be considered along with the other alternatives. If employed, however, the extreme position of this viewpoint should not be left out of sight since a decision it can produce can be only overcautioned and very careful, which may not always be sound. Imagine a military commander who makes all his decisions working from a hypothesis that his opponent is extraordinarily intelligent, shrewd and resourceful to counteract every time immediately and in a most unpleasant

[6] In more detail various approaches to choosing the decision alternatives under uncertainty are considered in Chapter 8.

way. Hardly would a strategy of this commander be a success. On the contrary, in any real situation, the effort of decision maker should be devoted to revealing where the weak points of the opponent lie so as to use the information in inventing a deceptive strategy.

A spirit of pessimism is even less appropriate in the situations where the decision maker is not opposed by any enemy forces. The "pessimistic" computations should always be corrected by a sound amount of optimism. The other extreme viewpoint, that of blind optimism, might be as dangerous, yet some risk must be involved in any process of decision making. We should not overlook also that any decision made under unfavorable uncertainty is bound to be bad and therefore hardly deserves to be substantiated with delicate and laborious computations. One should better think of where to collect the lacking information. Here all the ways are good if only they clarify the situation.

In this connection we are bound to mention a rather original approach which, though not exciting lovely feelings in pure mathematicians, is useful and at times the solely possible. We speak of *judgemental probability assessment*. It is often employed in problems where forecasting is to be made under unfavourable uncertainty as, say, in futurology. The idea of the approach is roughly in the following. A team of experts is recruited to answer a question, for example, to call the term of an invention, or assess the probability of an event. The collected answers undergo statistical treatment, as it were, and although the results retain the subjective nature, they do it in a lesser extent than if a single expert were interrogated. Similar assessments of probability may be applied in studies under uncertainty. Each of the experts assesses the likelihood of various alternative conditions $\xi$ assigning them some subjective

probabilities. Although each of the experts tends to introduce into the assessment his own bias, after averaging the result is more objective and useful (by the way, the estimates of real expertise disperse not so strongly as one might expect). In this manner a problem involving an unfavorable uncertainty reduces, as it were, to an ordinary stochastic problem. The conclusions drawn in this section should, certainly, not be overemphasized; if added to other results, yielded by other approaches, they may, however, prove valuable in selection of an alternative.[7]

Finally we will make a general remark. Whatever we have done trying to substantiate a decision selected under uncertainty, an element of guess still remains. Therefore we ought not to make too higher a demand to thus produced decisions. Instead of pointing out a sole, explicitly optimal (from some viewpoint) decision, it would be a better approach to evaluate a region of feasible decisions which are if only insignificantly inferior to the others at whatever conditions (viewpoint) we consider them. Within this region we (the researchers) must recommend management to make its final choice. Displaying the decision making recommendations, the researcher must simultaneously convey the viewpoints on which these recommendations are based.

## 2.3 Multiobjective Problems

Inspite of substantial difficulties offered by uncertainty, the situations we have considered so far can be

[7] Judgement probability assessment is implemented in other arrangements as the simplest one outlined above with each expert expressing his opinion independently of the others; at times joint assessment is employed (brainstorming session).

regarded as most simple, since the objective or measure of effectiveness to be maximized (minimized) has been sole and obvious in each of them. Unfortunately, such problems are not a frequent practical occasion, they can be presumably met in dealing with small-scale and moderate-consequence ventures. In studies of large-scale complex actions which involve different interests of management and community, the performance cannot be as a rule completely described by a sole measure so that additional measures have to be designed. Owing to the diversity of objectives the problems of the type are termed *multiobjective*.

As an example of such a problem consider a defence plan for an important facility to protect it from air attacks. We have at our disposal some antiaircraft weapons that must be rationally placed around the facility, arranged for mutual operation, have the targets distributed among them and adequate stock supply provided, and so forth. Assume that each of the enemy aircraft participating in a raid carries a powerful weapon which if hit home completely demolishes the facility. The primary objective of our operation is then to battle the aircraft off the facility to prevent any of them coming at an aiming point. Naturally a measure of effectiveness in this case will be built around the probability that none of the aircraft will break through our defence screen. Is it, however, the sole important objective? Certainly, not. The probability of rejection being equal, we would prefer a decision that cares for a larger count of enemy planes be shot down. Whence another measure, the average number of eliminated (hit) planes, which we also would like to maximize. Besides, we are not indifferent to our own losses—another measure we would like

to minimize. We might also wish that the amount of expended ammunition be minimal, and so on.

Another example, this time from the production planning field. Planning or reorganizing a production, management usually has to deal with several aspects. On the one hand, it would be desirable to maximize the output; on the other hand, the net profit must be made as large as possible. As for production costs, they must be least; while the labour productivity must be highest. If thought over a little more, the problem can produce additional objectives.

The multitude of effectiveness measures, of which ones would like to be driven to a maximum, others to a minimum, is characteristic of any complex OR problem of value. We suggest the reader to exercise the acquired skill devising measures to assess the performance of a bus company. Consider which of the measures, from your point of view, is superior (most intimately connected with the objective); place the rest in the order of descending importance. This is a good example to convince that (a) none of the measures can be selected as the sole one, and (b) establishing a system of measures is not so simple a problem after all. Both the measures and their priority depend on whose interests are preferred when a decision is optimized.

Resuming, we typically found a variety of objectives in a large-scale venture, with the associated quantitative performance indexes, $W_1$, $W_2$, . . ., $W_n$, some of which are desired to be maximized, and the others minimized.

The question now is whether there exists a decision such that can satisfy all these requirements. A frank answer to it is: no. A decision that maximizes one of the measures is as a rule unable either to minimize or to maximize the others. Therefore a formulation

such as 'to produce maximal effect at minimal expenditures' is nothing but a rhetorical phrase and must be discarded in any serious analysis.

How would we manage then if we would have to assess the effectiveness of an action by several measures?

Nonexperts in operations research would tend to produce a single performance criterion for such a multiobjective problem. They would devise a function of all the measures involved and consider it as a joint, generalized measure designed to optimize a sought solution. Often this measure is materialized as a fraction with the variables desired to be increased in the numerator, and those to be decreased in the denominator. For instance, productivity and profit are above the line, and expense below.

This approach cannot be recommended because it is based on an implicit assumption that a shortcoming of one of the measures may be made up at the expense of another; low output, say, at the expense of low cost. As a rule, this assumption does not hold true.

Recall the person's quality self-assessment criterion that was once jokingly devised by the famous Russian novelist Leo Tolstoy. He put it as a fraction with actual merits of a person in the numerator, and the self assessment in the denominator. At first sight, the criterion might seem logically solid. However, imagine a person almost devoid of merits and of nil opinion of himself. The criterion would judge him as one of infinitely high value which could hardly be agreed with.

Similar paradoxical conclusions can not infrequently be drawn from a rational measure of effectiveness in which all the factors that contribute to advantage are put in the numerator, and those to disadvantage in the denominator.

Another, a bit more sophisticated generalized measure of effectiveness in frequent use is drawn up as a weighted sum of individual measures each of which, $W_i$, enters with its own weighting coefficient $a_i$ reflecting the importance of a related measure:

$$W = a_1W_1 + a_2W_2 + \ldots \qquad (2.3\text{-}1)$$

Those measures which are desired to be increased come in with positive weights, those to decrease with negative.

For arbitrary weights $a_1, a_2, \ldots$ this approach is nowhere better than the previous, if only the generalized criterion this time does not go to infinity. Its advocates refer to the inherent analogy with that how a human working out a compromise mentally weighs all for and against assigning larger weights to more important factors. This similarity is true only in part since the weigting coefficients introducing various factors are not constant but vary accordingly as the situation alters.

Illustrate the statement with an everyday life example. A clerk being out for the office considers his route to arrive in time. Which transportation will be most suitable? Streetcars go in short intervals, but run long; buses run faster, but arrive at longer intervals. He may, certainly, take a taxi, but this will be the most expensive way. Or another tack: cover a part of the distance by underground and then take a taxi. Yet there may happen to be no cars at the taxi-rank and then he will have to walk the last part of the distance, jeopardizing to be late more than if he took a bus. How will he behave?

We face a typical (intentionally simplified) OR problem with two measures. The first is the mean expected time of being above schedule, $T$, which is

desired to be minimal. The second is the expected fare, $F$, which must also be kept at a minimum. The two requirements contradict each other, therefore, to make them compatible the clerk should arrive at a compromise between them. He may be weighing subconsciously all pros and cons employing somewhat of the generalized measure kind

$$W = a_1 T + a_2 F \Rightarrow \min \qquad (2.3\text{-}2)$$

The trouble though is that weighting coefficients $a_1$ and $a_2$ cannot be deemed constant. They depend on both the variables, $T$ and $F$, and the situation. If, for example, the clerk has been reprimanded recently for being late, the coefficient for $T$ will, likely, be increased, while the coefficient at $F$ will, probably, be increasing away from a payday. If he assigns (as is usually done) $a_1$ and $a_2$ arbitrarily, then the resultant "optimal" decision will be as arbitrary in its essence.

This example shows a typical trick for the situations of the sort—transfer of arbitrariness from one instance into another. A simple choice of a tradeoff decision by mental counterplaying all pros and cons for each alternative might have seemed arbitrary and not scientific enough, whereas manipulating a formula involving coefficients $a_1$ and $a_2$ (let alone their being assigned arbitrarily) looked more of a science. As a matter of fact, there is none of it in this manipulating either.

We ought to abandon our hope for getting rid completely of any subjective bias when choosing a decision alternative. Even in the simplest, one-measure problems it will inevitably be present, emerging if only in the choice of measure of effectiveness or establishing a mathematical model. The bias is even more inevi-

table in a multiobjective problem. Seldom, though they might be, the situations turn up which make it clear immediately after evaluating the measures for all alternatives which of them to choose. If an alternative decision is superior to the others in all measures, then obviously, this one is to be preferred. More often though the studies are met in which the situation cannot be made as plain at a first attempt; the measures usually tend to divert in opposite directions. Therefore it would be worthwhile to conduct additional analysis, invoking, possibly, even the expressions of (2.3-1) type though treating their implications critically rather than without reserve.

Helpless, as the considered situations might suggest it is, mathematical tooling is nevertheless of value in two aspects. First, it helps solving primal OR problems, i.e. calculating for any decision $x$ the relevant values of measures of effectiveness $W_1$, $W_2$, . . ., however many they might count (incidentally, multiobjective nature is not a hindrance for primal problems). Second and more important, it helps screen off those decisions of the set $X$ of possible decisions which are inferior to their counterparts in all measures.

The procedure of screening is fairly simple. Consider a multiobjective problem of $k$ measures, $W_1$, $W_2$, . . ., $W_k$. To simplify the discussion, assume all these quantities are to be maximized (changing signs, as will be recalled, is enough to make the maximum of a minimum). Let in the set of possible solutions be two $x_1$ and $x_2$ such that all $W_1$, $W_2$, . . ., $W_k$ for $x_1$ are equal or more than the respective criteria for the second solution, i.e. at least one of the measures assumes a larger value. Obviously then, there is no longer any sense to keep $x_2$ in $X$ as it is dominated by $x_1$. We

reject $x_2$ as noncompetitive and go over comparing all the other decisions through all the measures. The set $X$ may be drastically reduced as a result of this compare-reject procedure and retain only those (efficient) neither of which dominates the others.

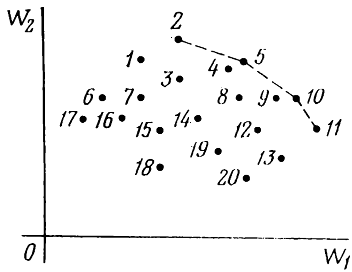


**Fig. 2.3-1**

To better illustrate how such decision can be evaluated, consider an example having two measures each of which is to be maximized. The set $X$ consists of a finite number of possible alternatives, $x_1, x_2, \ldots, x_n$. To each of them we may put in correspondence the relevant values of the measures $W_1$ and $W_2$. We will plot these values as points at a $W_1$ versus $W_2$ plane, bearing the numbers of the decisions (see Fig. 2.3-1).

As we can observe, of all the set $X$ we can refer to as efficient only the decisions $x_2$, $x_5$, $x_{10}$, and $x_{11}$ bordering from right and top the possible options area (dashed line in Fig. 2.3-1). For any other decision, there exists at least one such that dominates it either in $W_1$ or $W_2$ or both. It is only for the decisions on the upper boundary that no dominating decision exists.

Now that we have evaluated the efficient decisions out of the whole set of those possible, we may confine

our further search within the new "efficient" set. Figure 2.3-1 depicts it as consisting of four decisions, $x_2$, $x_5$, $x_{10}$, and $x_{11}$. Decision $x_{11}$ is seen to be the best in $W_1$, while $x_2$ the best in $W_2$. It is up to the decision maker to choose the alternative that would satisfy him in both criteria.

In a similar manner the efficient set is constructed when the measures are more than two (when more than three, a graphical representation can no longer be visualized, although geometrically a problem may be treated essentially along the same lines). The efficient set is easier to deal with than the set X. The final decision making is as ever left to management since what is required at this stage is to arrive at a trade-off decision which might be nonoptimal and nevertheless suit in several aspects; it will be made in an informal fashion by a decision maker bearing the responsibility for the final choice.

The procedure of selecting an alternative solution, if repeated several times, can yet form a basis to work out certain intuitive rules (called heuristics) with which a computer may proceed arriving at a final solution by its own. This approach received accordingly a name of 'heuristic programming'. It is essentially as follows. Assume that an expert or, still better, a group of experts having many times selected trade-off solution to a multiple alternative OR problem yields a series of estimates for various conditions $\alpha$. With these estimates at hand and enough time one can, say, arrive at a reasonable choice of weights $a_1, a_2, \ldots, a_n$ in expression (2.3-1), which generally depend on both $\alpha$ and measures $W_1, W_2, \ldots, W_n$ proper. A generalized criterion built with the weights may then be conveyed to a computer authorizing it to choose a solution by itself. The heuristic programming

approach has at times to be pursued when there is virtually no time for thinking a compromise decision over as, say, in a combat operation. It is also employed when an automatic control system is established in full charge of a process or system.

The computer also proves of value when the analyst has an ample time to come to conclusion by himself. He sits at the console and via the keyboard instructs the computer to calculate the values of measures $W_1, W_2, \ldots, W_n$. With these values on the display, he can critically assess the situation and appropriately correct the weighting coefficients, or other parameters of the control algorithm.

An often applied strategy of reducing the number of objectives in a multiple objective problem relies on evaluating a major measure, say, $W_1$, and driving it to a maximum, while imposing some constraints on the other measures, $W_2, \ldots, W_n$, requiring that they be not lower than specified values $w_2, \ldots, w_n$. For instance, in the optimization of production planning, management may require that the profit be maximized, items variety target be attained, and the costs of production be not above those specified. This approach translates all the measures, lest a major one, into the rank of given conditions $\alpha$. A certain freedom in assigning the margins $w_2, \ldots, w_n$ still persists so that the analyst may introduce the relevant corrections interactively gauging the problem via a computer.

Another way of arriving at a tradeoff solution is by means of what we would call successful concessions. Let the measures be ordered in decreasing priority. First, we seek a solution which maximizes the first (preemptive priority) measure so that $W_1 = W_1^*$. Next, considering low accuracy of initial data and the relevant practical environment we may assign some

concession $\Delta W_1$ which we can afford in stepping $W_1$ back in order to maximize the second measure $W_2$. With this limitation on $W_1$ requiring it be not lower than $W_1^* - \Delta W_1$, we seek a solution which maximizes $W_2$. The following step will obviously be an assignment of a concession in $W_2$ at the expense of which $W_3$ can be maximized, and so forth. The advantage side of this approach is in that it immediately discloses what should be the amount of tradeoff in one measure to gain in another and how large will be the winning.

Whatever stated, the multiple objective problem, being ill-structured, cannot be handled mathematically to a final point; the final decision is always left up to the decision maker. The analyst, or OR task group, supplies the manager with the data disclosing the advantages and disadvantages of each alternative solution in order to facilitate his decision towards more objectively substantiated preference.

* * *

To conclude, a few words are in order on the systems theory approach to decision making problems.

Now that scales and complexities of operations increase dramatically, ever pressing become the problems of optimal control concerned with large, dynamically interconnected systems involving a great number of units and subsystems organized into hierarchial structures. The principle can be exemplified by a large corporation which includes relatively self-governed subsidiaries with their plants and factories which in turn can be subdivided into shops and facilities. Trying to optimize the performance of one element of such a complex system, the manager must not overlook the interconnections that exist between the

units at various levels of the hierarchial structure. The existing influence implies that a segment cannot be optimized in isolation from the entire system.

Even within a smaller system, a plant, a sharp increase of production output as a result of an optimization is only a half of a success. If the manufactured items pile up—better if on the shelves, worse if in the yard—because the transportation facilities are not ready to carry them out, the inevitable losses may offset the gain due to the optimized production boost.

If we go further and suppose that the carriers were in abundance and there were no problems about transportation of the goods, however, the market stand had not been consulted so that the delivered surplus commodities swiftly diminished in price along with the other brought merchandise, it certainly would inflict losses—again resulted from "local" planning.

What sort of planning should then be recommended? It certainly must not be an all-inclusive, tightly-composed brain work, governed hierarchially upside down and controlling all up to the smallest thing in a shop. This is neither possible, nor worth striving at. A sound control for a sophisticated hierarchy should be organized so that each command from an upper level downward be given in general terms providing some freedom for local initiative, though, the objectives must be formulated in such a way that working towards them a unit performs in accord with the interest of upper levels and the system as a whole.

This is obviously easier said than done. A mathematical theory of large hierarchially structured systems is now still under development. The effort is devoted to creating mathematical tools which would appropriately describe such systems and decompose them into smaller subsystems or elements more suitable for

studies, yet so far efficient methods of control for such systems are still wanted. Practically, the 'systems approach' in the OR field is materialized as a treatment of a subsystem being optimized as a part of a larger system within which the subunit undergoes an evaluation of the influence it may impose on the performance of the whole system.

# Chapter 3

# LINEAR PROGRAMMING

## 3.1 Linear Programming Problems

In the previous chapters we considered mainly the methodology of operations research, studied various types of problems, evaluated ways to their solution, etc., while putting aside the subject of mathematical techniques needed for that purpose. This and ensuing chapters will shortly discuss some of these techniques widely employed in OR studies. We will not go into the details of their implementation, placing main emphasis on their basic structure.

We have already mentioned the simplest problems in which the measure of effectiveness (objective function), $W$, explicitly evolves from the orientation of the operation at hand and the conditions of the operation are specified beforehand (deterministic problem). For this class of problems the objective function depends upon two sets of parameters only, namely, defined conditions $\alpha$, and controllable (decision) variables $x$:

$$W = W(\alpha, x) \tag{3.1-1}$$

It is important to recall that the specified conditions $\alpha$ contain the constraints imposed on the decision variables.

Let a solution to our problem be a set of $n$ decision variables $x_1, x_2, \ldots, x_n$, or an $n$-tuple,

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]$$

It is required to find such values of **x** which maximize (or minimize), i.e. yield an optimum for, the objective function $W$.

The problems seeking the values of parameters that optimize the function subject to the constraints imposed on the arguments have come to be known as *problems of mathematical programming.*[1] They differ in complexity which is decided by (i) the type of the function relating $W$ with decision variables, (ii) the dimension of problem, i.e. the number of decision variables $x_1, \ldots, x_n$, and (iii) the type and number of constraints imposed on the variables.

The simplest (and best developed) problems of mathematical programming are the so-called *linear programming* (*LP*) *problems.* They are interesting in that (a) their measure of effectiveness (called mainly 'objective function' for this class of problems) depends linearly on the decision variables $x_1, \ldots, x_n$, and (b) the constraints imposed on the variables assume the form of linear equations or inequalities in $x_1, \ldots$ $\ldots, x_n$.

These are recurrent problems in practical applications, for example in resource allocation, production planning, transportation scheduling, to name only a few. This is only natural, as in many practical problems incomes and expenditures are linear functions of the amounts of acquired or utilized materials. For example, the total cost of a lot of goods varies in direct proportion with the number of items in the lot; transportation payments are again effected in direct proportion to carried weights—the examples may queue at length.

[1] The word programming does not refer to the programming of a computer; it means the programming or allocation of items or entities.

Of course, we are far from claiming all the practical relationships being linear; we would content ourselves with a modest statement that linear, or close, dependencies are frequent occasions—a valuable fact, too.

To get a better sense of the matter, consider several LP problems.

**A blending (diet) problem.** A feed mix for livestock at an animal farm is composed to meet some minimum daily calorie requirements of protein, carbohydrate, and fat. For simplicity assume that only four food-stuffs are available, $F_1$, $F_2$, $F_3$, and $F_4$. The cost of unit of each food is respectively $c_1$, $c_2$, $c_3$, and $c_4$. The calorie content of each food with respect to protein, fat, and carbohydrate on per unit basis is shown in Table 3.1-1 by symbol $a_{ij}$, where $i$ stands for the number of food-stuff, hence $i = 1, 2, 3, 4$; and $j$ stands for the number of nutrient: protein $= 1$, carbohydrate $= 2$, fat $= 3$.

Table 3.1-1

| Food | Protein | Carbohydrate | Fat | Cost |
|---|---|---|---|---|
| $F_1$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $c_1$ |
| $F_2$ | $a_{21}$ | $a_{22}$ | $a_{23}$ | $c_2$ |
| $F_3$ | $a_{31}$ | $a_{32}$ | $a_{33}$ | $c_3$ |
| $F_4$ | $a_{41}$ | $a_{42}$ | $a_{43}$ | $c_4$ |

The problem is to blend a mix, that is, to assign an amount to each food-stuff component, such that the protein content of mix be not less than $b_1$ units, carbohydrates not less than $b_2$ units, fat not less than $b_3$ units, and keep it at a minimum cost.

Let us build a mathematical model. This time the procedure will be rather simple. Denote as $x_1$, $x_2$, $x_3$,

and $x_4$ the amounts of each food-stuff in the mix. The objective we are going to minimize is the cost of the mix, $C$, which linearly depends on the controllable variables $x_1$, $x_2$, $x_3$, and $x_4$:

$$C = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

or in a more concise form

$$C = \sum_{i=1}^{4} c_i x_i \tag{3.1-2}$$

Thus, we have obtained the objective equation and it is linear. Next we write formulas for the constraints of protein, carbohydrate, and fat content. Considering that $x_1$ units of food $F_1$ contain $a_{11}x_1$ units of protein, $x_2$ units of food $F_2$ contain $a_{21}x_2$ units of protein, and so on, we arrive at three inequalities

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 &\geqslant b_1 \\ a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 &\geqslant b_2 \\ a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4 &\geqslant b_3 \end{aligned} \tag{3.1-3}$$

constraining the variations of decision variables $x_1$, $x_2$, $x_3$, and $x_4$.

The problem reduces thereby to the following: find nonnegative values of decision variables $x_1$, $x_2$, $x_3$, and $x_4$ such that satisfy the constraint inequalities (3.1-3) and simultaneously minimize the linear objective function of these variables:

$$C = \sum_{i=1}^{4} c_i x_i \Rightarrow \min$$

What we have obtained is a typical problem of linear programming. We postpone the discussion of its solution methods until later to illustrate three more problems of the kind.

**An inventory scheduling problem.** A factory is set up to produce three products $P_1$, $P_2$, and $P_3$. The company that controls the factory has a contract to supply not less than $b_1$ items of $P_1$, not less than $b_2$ items of $P_2$, and not less than $b_3$ items of $P_3$. The factory may manufacture still higher numbers of the products, but the demand bounds them from above: not more, respectively, than $\beta_1$, $\beta_2$, and $\beta_3$ items each. Each product requires for its manufacture four types of raw material $m_1$, $m_2$, $m_3$, and $m_4$. The stocks of each material are limited by amounts of $\gamma_1$, $\gamma_2$, $\gamma_3$, and $\gamma_4$ units. To produce an item of product $P_j$ the factory needs $a_{ij}$ units of raw material $m_i$ ($i = 1, 2, 3, 4$). The first subscript, $i$, is seen to refer to the product, and the second, $j$, to the raw material. The values $a_{ij}$ are summarized in Table 3.1-2.

Table 3.1-2

| Raw materials | Products | | |
|---|---|---|---|
| | $P_1$ | $P_2$ | $P_3$ |
| $m_1$ | $a_{11}$ | $a_{21}$ | $a_{31}$ |
| $m_2$ | $a_{12}$ | $a_{22}$ | $a_{32}$ |
| $m_3$ | $a_{13}$ | $a_{23}$ | $a_{33}$ |
| $m_4$ | $a_{14}$ | $a_{24}$ | $a_{34}$ |

The company realizes a profit of $c_1$ on each item of $P_1$, $c_2$ on each item of $P_2$, and $c_3$ on each item of $P_3$. How should the manufacturing of the products be arranged subject to the constraints placed by the contract and demand and so as to obtain the largest possible profit?

Let us formulate the problem in the form of a linear program. The controllable variables will be $x_1$, $x_2$,

and $x_3$, which are the numbers of items of $P_1$, $P_2$, and $P_3$ which the factory will produce. The necessity to meet the contract obligations will be presented by three constraint inequalities:

$$x_1 \geqslant b_1, \quad x_2 \geqslant b_2, \quad x_3 \geqslant b_3 \tag{3.1-4}$$

Three more inequalities appear due to the restrictions of demand:

$$x_1 \leqslant \beta_1, \quad x_2 \leqslant \beta_2, \quad x_3 \leqslant \beta_3 \tag{3.1-5}$$

Besides, the factory must not run out of raw material stocks. Accordingly, for four types of material we get four constraint inequalities:

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 + a_{31}x_3 &\leqslant \gamma_1 \\ a_{12}x_1 + a_{22}x_2 + a_{32}x_3 &\leqslant \gamma_2 \\ a_{13}x_1 + a_{23}x_2 + a_{33}x_3 &\leqslant \gamma_3 \\ a_{14}x_1 + a_{24}x_2 + a_{34}x_3 &\leqslant \gamma_4 \end{aligned} \tag{3.1-6}$$

The profit realized on a production schedule $(x_1, x_2, x_3)$ will be:

$$L = c_1x_1 + c_2x_2 + c_3x_3 \tag{3.1-7}$$

Thus we have obtained again an LP problem: find (choose) nonnegative values of variables $x_1$, $x_2$, and $x_3$ such that satisfy the constraint inequalities (3.1-4), (3.1-5) and (3.1-6) and maximize the linear function of these variables:

$$L = c_1x_1 + c_2x_2 + c_3x_3 \Rightarrow \max \tag{3.1-8}$$

This problem is closely akin to the previous one, the only difference being that this time constraint inequalities are more in number and the objective is maximized rather than minimized (we already know, however, that the latter discrepancy is insignificant).

The next problem prepares the constraints of still another type.

**A production scheduling problem.** A weaving mill manufactures three grades of cloth, $G_1$, $G_2$, and $G_3$, on two types of looms of which there are $N_1$ machines of type 1 and $N_2$ machines of type 2. The looms differ in production rate having various output for different cloths. The respective rates, $a_{ij}$, are summarized in Table 3.1-3 where $i$ refers to the type of machine and $j$ to the grade of cloth.

Table 3.1-3

| Loom type | Cloth | | |
|---|---|---|---|
| | $G_1$ | $G_2$ | $G_3$ |
| 1 | $a_{11}$ | $a_{12}$ | $a_{13}$ |
| 2 | $a_{21}$ | $a_{22}$ | $a_{23}$ |

The profit for each meter of cloth $G_1$ is $c_1$, for a meter of $G_2$ is $c_2$, and for a meter of $G_3$ is $c_3$.

To fill an order, the mill must produce monthly not less than $b_1$, $b_2$, and $b_3$ meters of each cloth, respectively. The warehouse capacity, however, demands that the produced amounts must not exceed, respectively, $\beta_1$, $\beta_2$, and $\beta_3$ meters. Besides, the operating conditions necessitate that all the machines be loaded.

The problem is to assign the machines to the production order so as to realize the largest monthly profit.

At first sight, what is formulated is a tween problem to the previous one. The temptation is very strong to denote the amounts of cloths in a production schedule by $x_1$, $x_2$, and $x_3$ and maximize the total

profit $c_1x_1 + c_2x_2 + c_3x_3$. Yet we should stop here and ask ourselves: where are the capacities of the looms? Upon a little thought it will be obvious that this time the decision variables are the numbers of machines of type 1 and type 2 employed in the manufacture of each cloth rather than the amounts of the cloths. For convenience we denote the decision variables by the letter $x$ with two variable indices, the first of which refers to a type of machine, and the second to a cloth. All in all, it yields six variables.

$$\begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{matrix} \tag{3.1-9}$$

where, say, $x_{11}$ stands for the number of type 1 machines employed in the manufacture of grade $G_1$ cloth.

We have come to formulating one more problem of linear programming. First, put down the constraints imposed on the decision variables $x_{ij}$. The major of them are the restrictions set by the order:

$$\begin{aligned} a_{11}x_{11} + a_{21}x_{21} &\geqslant b_1 \\ a_{12}x_{12} + a_{22}x_{22} &\geqslant b_2 \\ a_{13}x_{13} + a_{23}x_{23} &\geqslant b_3 \end{aligned} \tag{3.1-10}$$

This done, come over to satisfy the warehouse limitations which yields three more inequalities:

$$\begin{aligned} a_{11}x_{11} + a_{21}x_{21} &\leqslant \beta_1 \\ a_{12}x_{12} + a_{22}x_{22} &\leqslant \beta_2 \\ a_{13}x_{13} + a_{23}x_{23} &\leqslant \beta_3 \end{aligned} \tag{3.1-11}$$

Now is the turn for writing the conditions that all machines participate in the manufacture. Recall that the total number of the looms of type 1 employed in the manufacture must be $N_1$, and the machines of

type 2 amount to $N_2$. Whence these two equalities:

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= N_1 \\ x_{21} + x_{22} + x_{23} &= N_2 \end{aligned} \tag{3.1-12}$$

Finally, put down the total profit produced by all the cloths. The total amount of cloth $G_1$ manufactured, $a_{11}x_{11} + a_{21}x_{21}$, produces the profit $c_1\ (a_{11}x_{11} + a_{21}x_{21})$. The same type of reasoning yields, for a production schedule (3.1-9), the total profit of the mill for a month:

$$L = c_1\ (a_{11}x_{11} + a_{21}x_{21}) + c_2\ (a_{12}x_{12} + a_{22}x_{22}) + c_3\ (a_{13}x_{13} + a_{23}x_{23})$$

or in a concise form

$$L = \sum_{j=1}^{3} c_j \sum_{i=1}^{2} a_{ij}x_{ij} \tag{3.1-13}$$

This linear function of six arguments is to be maximized, i.e. $L \Rightarrow \max$.

In words, this linear program may be spelled thus: find nonnegative values of variables $x_{11}, x_{12}, \ldots, x_{23}$ such that satisfy the constraint inequalities (3.1-10), (3.1-11), and equalities (3.1-12) and maximize the linear function of these decision variables (3.1-13). The total count of the problem constraints amounts to eight: six inequalities and two equations.

**A raw material allocation problem.** A large corporation has three production plants, $P_1$, $P_2$, and $P_3$ in demand each, respetively for $a_1$, $a_2$, and $a_3$ units of a raw material. Five sources variously distanced from the plants can supply this material at various transportation costs per unit tabulated in Table 3.1-4.

The capacities of the suppliers are limited by the available stocks so that the sources $S_1$, $S_2$, $S_3$, $S_4$,

and $S_5$ can supply not more than $b_1$, $b_2$, $b_3$, $b_4$, and $b_5$ units, respectively. It is required to design a shipment schedule, i.e. to allocate the amounts supplied by each source to each plant, such that the demands are satisfied at least possible transportation costs.

Table 3.1-4

| Plant | Supplier | | | | |
|---|---|---|---|---|---|
| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ |
| $P_1$ | $c_{11}$ | $c_{12}$ | $c_{13}$ | $c_{14}$ | $c_{15}$ |
| $P_2$ | $c_{21}$ | $c_{22}$ | $c_{23}$ | $c_{24}$ | $c_{25}$ |
| $P_3$ | $c_{31}$ | $c_{32}$ | $c_{33}$ | $c_{34}$ | $c_{35}$ |

Formulate again the LP problem. Let $x_{ij}$ denote the amount of material received by the $i$th plant from the $j$th source. In total, a schedule will involve 15 decision variables:

$$\begin{matrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \end{matrix} \tag{3.1-14}$$

The constraints on demand must state that each plant receives as much material as it demands, whence

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} + x_{15} &= a_1 \\ x_{21} + x_{22} + x_{23} + x_{24} + x_{25} &= a_2 \\ x_{31} + x_{32} + x_{33} + x_{34} + x_{35} &= a_3 \end{aligned} \tag{3.1-15}$$

Next put down the constraints evolving from the

capacities of the suppliers:

$$
\begin{aligned}
x_{11} + x_{21} + x_{31} &\leqslant b_1 \\
x_{12} + x_{22} + x_{32} &\leqslant b_2 \\
x_{13} + x_{23} + x_{33} &\leqslant b_3 \\
x_{14} + x_{24} + x_{34} &\leqslant b_4 \\
x_{15} + x_{25} + x_{35} &\leqslant b_5
\end{aligned}
\qquad (3.1\text{-}16)
$$

Finally come the total costs which we are to minimize. With reference to Table 3.1-4 we may put them as

$$L = \sum_{i=1}^{3} \sum_{j=1}^{5} c_{ij} x_{ij} \Rightarrow \min \qquad (3.1\text{-}17)$$

We again have obtained a linear program: find nonnegative values of variables $x_{ij}$ such that minimize the linear function (3.1-17) subject to the constraints (3.1-15) and (3.1-16).

Summing up, we have considered several typical LP problems. They have similar formulations and differ only in that in ones the objective must be maximized whereas in others it must be minimized; in some problems the constraints are all inequalities, in others they may be both inequalities and equations. There are LP problems where all constraints are equalities. These differences, though, are of minor importance since constraint inequalities may be readily transformed into equations and vice versa, as will be demonstrated in the ensuing section.

## 3.2 Moving to Algebraic Solution

Any LP problem may be cast into a standard form which within the framework of this text will be called the general LP problem.

It is stated as follows: find nonnegative values of variables $x_1$, $x_2$, . . ., $x_n$ such that satisfy the constraint equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \qquad (3.2\text{-}1)$$

and maximize the linear function of these variables:

$$L = c_1x_1 + c_2x_2 + \ldots + c_nx_n \Rightarrow \max \qquad (3.2\text{-}2)$$

Let us demonstrate how it can be done. First, recall that the case when $L$ must be minimized rather than maximized is easily handled by changing the sign of $L$ and maximizing $L' = -L$. Second, any constraint inequalities may be converted to equations by introducing additional variables, called 'slack variables' or simply 'slacks'. We demonstrate the procedure by an example.

Suppose that it is required to find nonnegative values of variables $x_1$, $x_2$, and $x_3$ such that satisfy the constraint inequalities

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &\geqslant 4 \\ x_1 - 2x_2 + 3x_3 &\leqslant 10 \end{aligned} \qquad (3.2\text{-}3)$$

and maximize the linear function of these variables:

$$L = 4x_1 - x_2 + 2x_3 \Rightarrow \max \qquad (3.2\text{-}4)$$

We start with providing inequalities (3.2-3) with zeroes on the right-hand side, namely,

$$\begin{aligned} 3x_1 + 2x_2 - x_3 - 4 &\geqslant 0 \\ -x_1 + 2x_2 - 3x_3 + 10 &\geqslant 0 \end{aligned} \qquad (3.2\text{-}5)$$

Denote the left-hand sides in (3.2-5) by $y_1$ and $y_2$:

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 - x_3 - 4 \\ y_2 &= -x_1 + 2x_2 - 3x_3 + 10 \end{aligned} \qquad (3.2\text{-}6)$$

The inequalities (3.2-5) are seen to secure the nonnegativity of $y_1$ and $y_2$. After this manipulating, the problem reformulates as follows. Find nonnegative values of variables $x_1$, $x_2$, $x_3$, $y_1$, and $y_2$ such that maximize the linear function of these variables subject to the constraint equations (3.2-6). The absence of the slack variables $y_1$ and $y_2$ from $L$ can be easily remedied by assuming that they enter it with zero coefficients. Thus we have arrived at the general LP problem. The transformation from the original inequality constraint formulation has been effected at the expense of increasing the number of variables by two (equal to the number of the former inequalities).

The reverse transformation is possible as well. Assume that we face a general LP problem with the equality constraints (3.2-1). Assume further that only $r \leqslant m$ of these $m$ equations are linearly independent[2]. Any standard text on linear algebra (see, for example, [4]) proves that $n$ variables $x_1, x_2, \ldots, x_n$ can be connected into at most $n$ linearly independent equations, so in general $r \leqslant n$. Another fact from algebra is that a system of $r$ independent equations in $n$ variables $x_1, x_2, \ldots, x_n$ can always be solved for some $r$ variables, termed *basic variables*, relative to the rest $k$

[2] Equations are referred to as linearly independent if none of them can be obtained from the others by multiplying them by scalars and summing up, i.e. none is the consequence of the others.

$= n - r$ variables, termed *nonbasic*. The resulted nonbasic variables can assume any values without violating the conditions (3.2-1). Consequently, to convert the constraint equations (3.2-1) into inequalities, it is sufficient to solve (3.2-1) for some $r$ variables chosen as basic, or those *in solution*, expressing them via the nonbasic variables. Recalling that all the variables must be nonnegative, put down the nonnegativity conditions in the form of constraint inequalities. Then forget, as it were, about the basic variables to manipulate by the nonbasic ones amounting to $k = n - r$. The objective function, $L$, must also be expressed through the nonbasic variables only, by substituting for the basic ones the respective expressions in nonbasic variables. As can be seen the translation from a general LP formulation to that with constraint inequalities reduces the number of variables by the number $r$ of independent constraint equations making up the constraining set in the general LP problem. We will not illustrate the procedure with examples, leaving the possibility of check to the reader.

Summing up, any LP problem can be cast into a general LP problem formulation. This book contains no detail treatment of its solution techniques. They are discussed at length in many special texts (consult, for example, [4, 5]) and numerous texts on operations research (e.g. [6, 7]). The subsequent section considers only general principles of how to evaluate whether or not a particular general LP problem is soluble and how its solution can be found. The relevant computational algorithms will not be treated; the interested reader may consult for them the mentioned texts, for example.

## 3.3 A Geometric Method of Solution

Consider the general LP problem which is formulated as follows: find nonnegative values of variables $x_1, x_2, \ldots, x_n$ such that maximize the linear function of these variables

$$L = c_1x_1 + c_2x_2 + \ldots + c_nx_n \Rightarrow \max \qquad (3.3\text{-}1)$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \qquad (3.3\text{-}2)$$

For simplicity we assume that all the conditions (3.3-2) are linearly independent, i.e. $r = m$.[3]

Any set of nonnegative values $(x_1, x_2, \ldots, x_n)$ satisfying the conditions (3.3-2) is called a *feasible solution* of the LP problem. A feasible solution at which the linear objective function (3.3-1) is optimum is an *optimal solution.* Hence, the problem is to find an optimal solution.

This problem, however, is not always soluble. It may happen so that the equations (3.3-2) are inconsistent. On the other hand, they may turn out consistent, but solely outside the area of nonnegative solutions, i. e. no collection of numbers $x_1 \geqslant 0, x_2 \geqslant 0, \ldots, x_n \geqslant 0$ exists such that meet the conditions (3.3-2). And, finally, feasible solutions may exist, but none of them be optimal because the function $L$

[3] If among the constraint equations (3.3-2) some are redundant, i.e. can be obtained from the others by linear transformations, this reveals automatically in solving the problem (refer to [4, 6]).

is not bounded from above in the area of feasible solutions. All these "hazards" might be solely prepared by thought of, or unnaturally formulated problems, though occasionally insoluble LP problems may result from insubstantial planning, say incomplete account of available resources.

To get a better insight into the principles underlying the general LP problem, we take up a geometrical approach. Let the number of structural constraints (equations) be two units smaller than the number of variables, i.e., $n - m = k = 2$. This particular case enables us to treat the problem graphically on plane.

As will be recalled, $m$ linearly independent equations (3.3-2) can always be solved for some $m$ basic variables, expressing them via the other, nonbasic variables, counting $n - m = 2$ in the considered case. Assume that the nonbasic variables are $x_1$ and $x_2$ (if they prove to the contrary, we may renumerate the variables) and the others, $x_3, x_4, \ldots, x_n$, are basic variables. Solving (3.3-2) for the basic variables yields

$$\begin{aligned} x_3 &= \alpha_{31}x_1 + \alpha_{32}x_2 + \beta_3 \\ x_4 &= \alpha_{41}x_1 + \alpha_{42}x_2 + \beta_4 \\ &\cdots\cdots\cdots \\ x_n &= \alpha_{n1}x_1 + \alpha_{n2}x_2 + \beta_n \end{aligned} \tag{3.3-3}$$

We will illustrate a graphic solution method in the $(x_1, x_2)$ coordinates. Since the variables $x_1$ and $x_2$ must be nonnegative, the area of the allowable values is the first quadrant, that is, above the axis $0x_1$, on which $x_2 = 0$, and to the right of the axis $Ox_2$, on which $x_1 = 0$. Figure 3.3-1 shows it as the region within the shaded boundaries.

We come now to constructing on the plane $x_1Ox_2$ the region of feasible solutions, or convincing ourselves

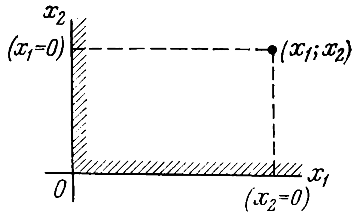


**Fig. 3.3-1**

that it does not exist. The basic variables $x_3$, $x_4$, . . . . . ., $x_n$ must be nonnegative as well as satisfy the equations (3.3-3). Each of these equations restricts the area of feasibility. To prove, assume in the first

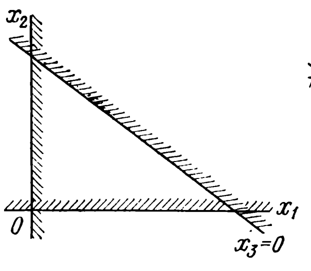


**Fig. 3.3-2**

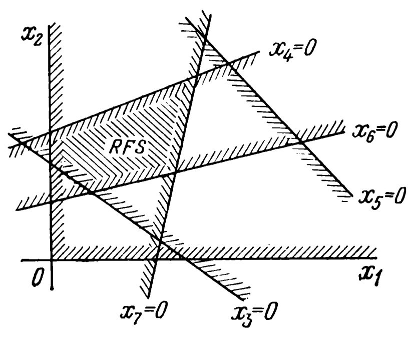


**Fig. 3.3-3**

equation of (3.3-3) $x_3 = 0$; this results in the equation of straight line

$$\alpha_{31}x_1 + \alpha_{32}x_2 + \beta_3 = 0$$

On this line, $x_3 = 0$; above it $x_3 > 0$, below $x_3 < 0$. We stripe the half-plane where $x_3 > 0$ (see Fig. 3.3-2). (To avoid unnecessary clutter, the shading of the region of feasibility is supressed to points near the boundary.) Now, the region of feasible solutions, or RFS for short, lies in the first quadrant, above the line $x_3 = 0$. Proceeding in exactly the same manner, we will graph all the other constraints (3.3-3). Boundary shading will show for each of them the half-plane where feasible solutions may be found.

In all, we have drawn $n$ lines: two axes of coordinates, $Ox_1$, and $Ox_2$, and $n - 2$ lines $x_3 = 0$, $x_4 = 0$, . . . . . ., $x_n = 0$. Each of them defines the feasible half-plane where feasible solution set must belong to. The part of the first quadrant which simultaneously belongs to all these half-planes is the sought *feasible solution area*. Figure 3.3-3 shows the graph with an existing region of feasibility, that is, the system (3.3-3) in this case has nonnegative solutions. Note in passing that these solutions are indefinitely many since any pair of nonbasic variables taken from the region of feasible solution satisfies the constraints along with the basic variables defined by it.

It may so happen that the region of feasible solution does not exist as a consequence of the equations (3.3-3) being inconsistent in the positive quadrant. This situation is illustrated in Fig. 3.3-4 showing no area that would lie simultaneously to the desired side of all lines. This implies that the respective LP problem has no solution.

Suppose that a region of feasibility exists and we have constructed it. Next step is to find an optimal solution in this area. To this end, we approach the condition (3.3-1), $L \Rightarrow \max$, graphically. Substituting the equations (3.3-3) into (3.3-1) yields $L$ in terms

of nonbasic variables $x_1$ and $x_2$. After collecting similar terms, we have

$$L = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_0 \tag{3.3-4}$$

where $\gamma_1$ and $\gamma_2$ are certain coefficients, and $\gamma_0$ is a free term which $L$ lacked originally, yet which might

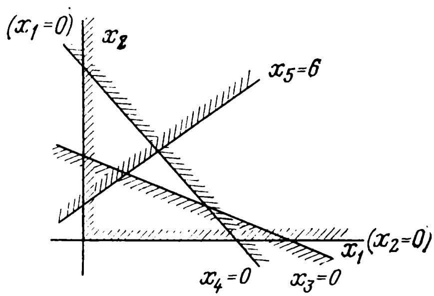


**Fig. 3.3-4**

appear upon the transformation to variables $x_1$ and $x_2$. We will, however, discard it immediately recalling that the linear homogeneous function

$$L' = \gamma_1 x_1 + \gamma_2 x_2 \tag{3.3-5}$$

achieves its maximum with the same values of $x_1$ and $x_2$.

Thus we ought to set out a graphical attack on $L'$. First we set $L' = 0$ and draw the resulting line $\gamma_1 x_1 + \gamma_2 x_2 = 0$ on the plane $x_1 O x_2$. This line—we shall term it "level line"—passes, obviously, through the origin. If we assign to $L'$ some values, the line will shift parallel to itself. Travelling in one direction it will increase $L'$, in the other decrease $L'$ in value.

Figure 3.3-5 indicates with arrows the direction of increase. This figure plots it upwards to the right,

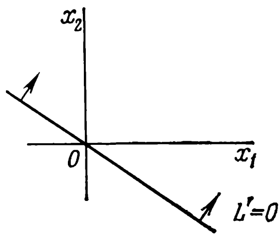


Fig. 3.3-5

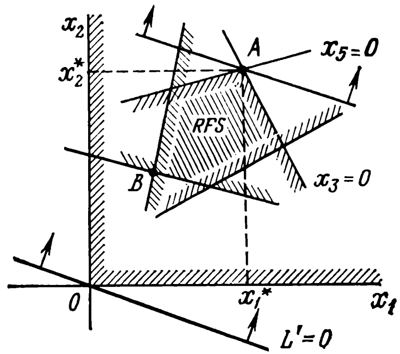


Fig. 3.3-6

but it might be to the contrary—the matter depends upon the coefficients $\gamma_1$ and $\gamma_2$. Now we plot the level line and the region of feasible solution (RFS) in one graph (see Fig. 3.3-6). Let us mentally shift the level line parallel to itself in the direction of the arrows (where $L'$ increases). Where will it reach a maximum? Obviously, it will be at point $A$, the extreme point of the RFS in the direction of increase.[4] The coordinates of this point $(x_1^*, x_2^*)$ yield the optimum values for the nonbasic variables which when substituted into (3.3-3) give the optimal values for the rest (basic) variables, i.e. $x_3^*, x_4^*, \ldots, x_n^*$. An important thing to note is that $L'$ attains its maximum at one of the vertices of the feasibility polygonal region where at least two basic variables turn to zero. The value of

[4] If the arrows pointed otherwise (downward to the left) the extreme point of the RFS would be point $B$.

zero might be assumed by more basic variables, should more straight lines $x_i = 0$ pass through the point $A$.

Can there be a situation with a feasibility region existing and an optimal solution nonexisting? Yes,

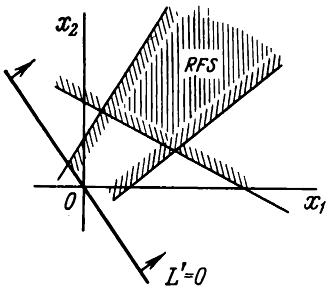


Fig. 3.3-7

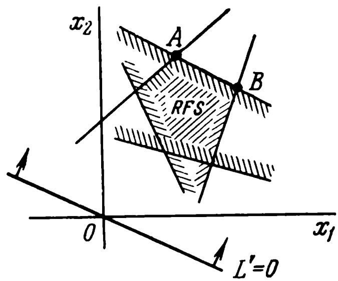


Fig. 3.3-8

it can if the function $L'$ (and consequently $L$, too) is not of bounded variation in the RFS. An example of such an abnormal situation with $L'$ not bounded from above is depicted in Fig. 3.3-7 (this misfortune is out of place, though, if a problem is reasonably posed).

Figure 3.3-6 graphs the problem for which an optimal solution exists and is unique. The case when an optimal solution exists but is not unique can be visualized graphically as, e.g. that in Fig. 3.3-8 illustrating a maximum of $L'$ being attained at the edge $AB$ parallel to the level line, rather than at a single point. This situation can be met in practice, but we should not worry about it. Anyway, $L'$ reaches its maximum at some of the feasible polygon vertices (whether $A$ or $B$ is insignificant), therefore the search for an optimal solution may be performed among the vertices of the RFS.

Having considered graphically the LP problem for which $n - m = 2$, we can now issue the following statement:

> an optimal solution (if any) is always attained at one of the vertices of the region of feasible solution, at which at least two of the variables $x_1, x_2, \ldots, x_n$ are zero.

Similar rule is proved to be valid for the case $n - m = k > 2$ as well (we can, however, no longer work it out graphically as we cannot draw in more than three dimensions). We content ourselves with solely formulating this rule.

> An optimal solution of a linear programming problem (if any) is attained at a set of values of variables $x_1, x_2, \ldots, x_n$ such that turns at least $k$ of these variables to zero, the others being nonnegative.

For $k = 2$, this set of values plots as a point on the plane, situated in one of the vertices of the feasible solution polygon. For $k = 3$, the set of feasible solutions is no longer a polygon, but a polyhedron; the optimal solution lies at one of its vertices. A graphical treatment cannot be effected for $k > 3$, but geometrical analogy retains its attractive convenience. As before, we will speak of a 'feasible solution polyhedron' in a $k$-dimensional space, and an optimal solution (if any) will again be sought at one of the vertices of this polyhedron, where at least $k$ variables are zero and the others nonnegative. To be concise, we shall term such a vertex a "basic point" and the relevant solution will be called a *basic solution.*

This geometrical approach gives rise to the idea underlying most of the solution techniques in linear programming—that of iterative search. To demonstrate, solve the equations (3.3-2) for some of the $m$ basic

variables and write them in terms of the rest $k$ nonbasic variables. Set these nonbasic variables equal to zero. If we are fortunate enough, this will yield a basic point. Compute the basic variables at zero-valued nonbasic variables. If all of them turn out nonnegative, then we are lucky to get a feasible (basic) solution at a first trial and have only to optimize it. If they are not, then this choice of basic and nonbasic variables fails to produce a feasible solution, i.e. the point lies beyond the RFS rather than on its boundary. A way out is to resolve the equations for another choice of basic variables in a non-arbitrary manner such that would get us closer to the feasibility region (for this purpose linear programming has developed special techniques, but we will not concentrate on them in this text).

Suppose that having repeated the procedure several times we have arrived at a basic solution. Half way there, but that's still not all. We need to check whether or not this solution is optimal. Let us express the function $L$ in terms of the latter set of nonbasic variables and then raise them from zero. If this only decreases the value of $L$, then the solution on hand is optimal. If not, then resolve the system of equations for other basic variables, again not in an arbitrary way but so as to near an optimal solution and not to leave the region of feasible solutions. For this purpose, too, linear programming has computational algorithms warranting that with each next resolving we will move closer to an optimal solution, or at least not away from it. We will not discuss these techniques here either. After a final number of such computational runs the objective will be attained and an optimal solution

found. In case an optimum does not exist, the implemented algorithm will demonstrate it.

The reader may ask whether the entire matter is that simple. Why should we have been all those subtle solution techniques about when all we need is to barely examine one by one all the possible combinations of $k$ nonbasic variables setting them equal to zero until finally an optimal solution is found?

For the simple problems where the number of variables is small, such an exhaustive search may indeed evaluate a solution and do this quickly enough. Applicational problems, however, not infrequently contain variables and imposed constraints amounting in hundreds and even thousands. For these problems a simple exhaustive method becomes unfeasible because the number of basic and nonbasic variable combinations rises too high. To illustrate, at $n = 30$ and $m = 10$ the number of possible combinations which yield nonbasic variables with basic variables is ${}_{30}C_{10} =$ 30 045 015, that is more than 30 million combinations! The problem though is not of a sophisticated variety.

The computational techniques developed for linear programming—the simplex method, the dual simplex method, and others, refer, for example, to [4, 7, 8]—seek for an optimal solution in direct iterative, rather than exhaustive, manner, appearing close to the solution with every step. Computer hardware manufacturers supply as a rule the relevant linear programming software so that a person facing an LP problem need not learn manual programming which for the problems of this class might be extremely monotonous and time consuming.

## 3.4 The Transportation Problem

The previous section has been concentrated on some general approaches to linear programming solution. Among the discussed family of problems, however, there are some which owing to their outstanding structure are amenable to more simple solution methods. We consider one of them, the *transportation problem*. The transportation problem requires the allocation of units located at a number of origins to a number of destinations in such a way that the allocation is an optimum (least cost or maximum profit). It is formulated as follows. The amounts of units available at each of $m$ origins (shipping sources) $A_i$ $(i = 1, \ldots, m)$ are $a_1, a_2, \ldots, a_m$ units, respectively. The amounts required at each of $n$ destinations $B_j$ are $b_1, b_2, \ldots, b_n$ units, respectively. These values are often referred to as *rim requirements*. The sums of all demands and supplies must be equal:

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{3.4-1}$$

The cost of transporting (or profit of allocating) one unit from origin $A_i$ to destination $B_j$ $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, n)$ is $c_{ij}$. All the numbers $c_{ij}$ are known; composed in a matrix form they may be written as

$$\begin{matrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{matrix} \tag{3.4-2}$$

For shortness, we will designate this matrix by the symbol $[c_{ij}]$.

The transportation costs are deemed to vary in direct proportion to the number of transported units.

It is required to design a transportation schedule (i.e. to allocate the origins, destinations and transported amounts) such that all the demands are met at a minimum cost.

Transforming the problem into the framework of the LP problem, denote by $x_{ij}$ the amount allocated from origin $A_i$ to destination $B_j$. The values $x_{ij}$ must be nonnegative. Together they form the matrix

$$\begin{array}{cccc} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{array} \tag{3.4-3}$$

which we will shortly denote by $[x_{ij}]$. We will refer to a set of values $[x_{ij}]$ as a *transportation schedule*, and to the variables $x_{ij}$ themselves as *allocations* or *shipments*. These nonnegative variables must satisfy the following constraints.

1. The total amount of supply allocated from each origin to all destinations must be equal to the amount available at the particular origin. This restriction gives $m$ constraint equations:

$$\begin{array}{c} x_{11} + x_{12} + \ldots + x_{1n} = a_1 \\ x_{21} + x_{22} + \ldots + x_{2n} = a_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{m1} + x_{m2} + \ldots + x_{mn} = a_m \end{array} \tag{3.4-4}$$

2. The total amount of supply delivered to each destination from all origins must be equal to the

particular demand. This yields $n$ constraint equations:

$$\begin{aligned} x_{11} + x_{21} + \ldots + x_{m1} &= b_1 \\ x_{12} + x_{22} + \ldots + x_{m2} &= b_2 \\ \ldots\ldots\ldots\ldots&\ldots \\ x_{1n} + x_{2n} + \ldots + x_{mn} &= b_n \end{aligned} \qquad (3.4\text{-}5)$$

3. The total transportation cost of all allocations, i.e. the sum of $x_{ij}$ multiplied by the respective costs $c_{ij}$ must be minimal

$$L = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \Rightarrow \min \qquad (3.4\text{-}6)$$

where the double summation operator implies that the summation is performed over all combinations of changing indices $i$ and $j$, that is, through all the origin/destination pairs.

We have stated a linear program with constraint equations (3.4-4) and (3.4-5) and linear objective function (3.4-6) which is to be minimized. It is outstanding in that all the coefficients in (3.4-4) and (3.4-5) are unities, the feature that lends the problem amenable to very simple solution methods. We will discuss some of them in what follows.

First of all notice that the constraints (3.4-4) and (3.4-5) are no longer linearly independent as their right-hand sides are related by the condition (3.4-1). The number of those linearly independent among equations (3.4-4) and (3.4-5) is therefore $m + n - 1$ rather than the total number of equations $m + n$. The total number of variables $x_{ij}$ in our problem is $m \times n$; whatever basic variables the equations (3.4-4) and (3.4-5) were solved for, their number would be exactly $m + n - 1$, while the number of nonbasic

variables

$$k = mn - (m + n - 1) = (m - 1)(n - 1)$$

As will be recalled, an optimal solution to an LP problem is attained at a vertex of the feasibility area, where at least $k$ variables assume the value of zero. For the case in question, an optimal shipment schedule requires that at least $(m - 1)(n - 1)$ shipments must vanish, that is, no shipment is effected from respective origins to destinations.

A shipment schedule will be termed feasible if it satisfies the constraints (3.4-4) and (3.4-5), i.e. all demands are satisfied and all supplies are exhausted. A feasible shipment schedule will be referred to as basic feasible schedule (solution) if it has at most $m + n - 1$ positive shipments (the other shipments are zeroes), otherwise it is *degenerate*. A schedule $[x_{ij}]$ will be termed optimal if among all feasible schedules it results in the lowest total transportation costs.

Owing to the special structure of the transportation problem its solution requires no lengthy recursive manipulations of the systems of equations. Instead, the search for optimal solution is performed within a table containing the data of the problem set up in a certain order. The entries to the table are origins and destinations, demands and supplies, transportation costs, and, if allocated, shipments positioned in the respective cells. A transportation table has $m$ rows and $n$ columns. In the right-hand corner of each cell we place the transportation cost $c_{ij}$ payed to transport a unit of commodity from $A_i$ to $B_j$. A shipment will be placed in the center of the cell. The cell corresponding to the shipment from $A_i$ to $B_j$ will be shortly denoted as $(i, j)$ shipping route.

An example of transportation table with $m = 4$ and $n = 5$ is given in Table 3.4-1 where the data of the problem has been entered but allocations have not been made.

Table 3.4-1

| Origin | Destination | | | | | Supply |
|---|---|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | |
| $A_1$ | 13 | 7 | 14 | 7 | 5 | 30 |
| $A_2$ | 11 | 8 | 12 | 6 | 8 | 48 |
| $A_3$ | 6 | 10 | 10 | 8 | 11 | 20 |
| $A_4$ | 14 | 8 | 10 | 10 | 15 | 30 |
| Demand | 18 | 27 | 42 | 15 | 26 | 128 |

First of all we should compose an initial (basic feasible) schedule. It may be readily effected by the *northwest-corner method*[5]. We will demonstrate it by

[5] Also referred to as the northwest-corner rule within the stepping-stone method with inspections. The rule states that the quantities shipped must begin in the upper left-hand corner of the table used in the method.—*Translator's note.*

means of the data in Table 3.4-1. The filling of the transportation table begins with the upper left-hand (northwest) corner, by satisfying the demand of destination $B_1$. We allocate the required 18 units from the supply of $A_1$. The remaining $30 - 18 = 12$ units after this shipment we allocate to destination $B_2$. Since its demand is not satisfied in full, we allocate the completing 15 units from the stocks of $A_2$. Using this procedure the table is filled with shipments $x_{ij}$ down to the lower right cell (see Table 3.4-2). The feasibility check for this schedule reveals that the rim requirements are satisfied, i.e. the sum of all shipments in a row is exactly the amount available at the respective origin, and the sum of allocations in a column equals the requirement of the respective destination. It means that all the demands are met and all the supplies are exhausted (the sum of supplies is equal to the total demand, 128 units entered in the lower right cell).

Henceforth we will fill into the cells only those shipments which are nonzero; the other cells are left unfilled. Let us verify now whether or not this initial solution is degenerate, i.e. whether the filled cells (with amounts to be shipped) are not too many. The number of unfilled cells in Table 3.4-2 is 12 which is exactly the figure $(m - 1)(n - 1) = 12$ to qualify as nondegenerate.[6]

Now is the time to test this schedule for optimality, i.e. to test whether its total transportation costs are minimal. It most likely will be not, since designing the transportation schedule we did not pay attention

[6] Another formula to test for degeneracy is $m + n - 1$, i.e. the number of filled cells must be the rim requirements (rows and columns) minus one for the solution to be not degenerate.— *Translator's note.*

Table 3.4-2

| Origin | Destination | | | | | Supply |
|---|---|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | |
| $A_1$ | 18 [13] | 12 [7] | [14] | [7] | [5] | 30 |
| $A_2$ | [11] | 15 [8] | 33 [12] | [6] | [8] | 48 |
| $A_3$ | [6] | [10] | 9 [10] | 11 [8] | [11] | 20 |
| $A_4$ | [14] | [8] | [10] | 4 [10] | 26 [15] | 30 |
| Demand | 18 | 27 | 42 | 15 | 26 | 128 |

to costs. The nonoptimality of this schedule can be evaluated easily. For example, it can be readily seen that it can be improved by having the lowest transportation costs associated with the filled cells, which implies moving the columns around. For example, we may cut the shipment in route (2,3) of high cost, but instead raise the allocation of low cost shipping route (2,4). To secure nondegeneracy in this movement from used into unused cells we have to empty one of the filled cells. How many units may we move through the closed system of routes (2,4) $\rightarrow$ (3,4) $\rightarrow$ (3,3) $\rightarrow$

$\rightarrow$ (2,3), increasing shipments in the odd cell routes of the set and decreasing in the even? Obviously, not more than 11 units, or else the shipping in cell (3,4) would be negative. As can be verified, the movement of allocations does not violate the rim requirements—the supplies and demands are balanced. Performing this shift yields an improved transportation schedule displayed in Table 3.4-3.

Table 3.4-3

| Origin | Destination | | | | | Supply |
|---|---|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | |
| $A_1$ | 18 [13] | 12 [7] | [14] | [7] | [5] | 30 |
| $A_2$ | [11] | 15 [8] | 22 [12] | 11 [6] | [8] | 48 |
| $A_3$ | [6] | [10] | 20 [10] | [8] | [11] | 20 |
| $A_4$ | [14] | [8] | [10] | 4 [10] | 26 [15] | 30 |
| Demand | 18 | 27 | 42 | 15 | 26 | 128 |

Consider now what are the cost savings with this new schedule. The total shipping costs of the schedule presented in Table 3.4-2 are $L_1 = 18 \times 13 + 12 \times 7 +$

$+ 15 \times 8 + 33 \times 12 + 9 \times 10 + 11 \times 8 + 4 \times 10 + 26 \times 15 = 1442$ whereas those in Table 3.4-3 amount to $L_2 = 18 \times 13 + 12 \times 7 + 15 \times 8 + 22 \times 12 + 11 \times 6 + 20 \times 10 + 4 \times 10 + 26 \times 15 = 1398$.

Thus we have lowered the total shipping costs by $1442 - 1398 = 44$ monetary units. The result, though, may have been predicted with no resort to the total shipping costs account. To prove, the algebraic sum of shipping costs of the moving route cells taken with a plus sign if the shipment of the cell increases and with a minus sign if decreases—called the net cost—amounts for the above example to $6 - 8 + 10 - 12 = -4$. It means that in moving one unit through a closed set of routes (the so-called evaluation method) the shipping costs diminish by four. We have moved 11 units which means that the costs must have been diminished by $11 \times 4 = 44$ monetary units, which is obviously the case.

We now may conclude that essentially all we need to optimize shipping schedule is to transfer shipments round closed systems of routes having a negative net cost.

Linear programming has the following statement proved for a nondegenerate shipping schedule.

> For each unfilled cell in a transportation table there exists a unique closed set of routes with one turning point in this cell and the others in filled cells.

Therefore, looking for cost saving paths of negative net cost we must evaluate those unused cells with low shipping costs. If such a cell is available, it should be included in an appropriately chosen closed set of routes. The next step is to evaluate the net cost of moving a unit through this system and if it turns out

negative, then we should move through the path as many units as possible without driving any shippings onto a negative level. The unused cell, consequently, becomes filled and one of the used cells becomes empty. This is obviously equivalent to resolving the system of equations for other basic variables with the difference that the tabular method performs it far easier.

Let us make another attempt at improving the transportation schedule given in Table 3.4-3. Suppose that we have chosen an attractive cheap route cell (1,5) worth 5. An obvious trend would be to raise the shipment in this cell at the expense of cutting shipments in some others (some routes would also have to ship more). Let us give an attentive treatment to Table 3.4-3 to evaluate a closed path whose first cell is (1,5) which is unfilled, and the other routes are used cells. Upon a little thought we arrive at the path: $(1,5) \to (4,5) \to (4,4) \to (2,4) \to (2,2) \to (1,2)$; completing it we again move to (1,5). The odd routes of the path are marked with a plus sign implying that the respective shippings increase, the even routes are marked with a minus sign indicating that they will ship less. The entire path is graphed in dashed lines in Table 3.4-4.

Compute the net cost of this path: $5 - 15 + 10 - 6 + 8 - 7 = -5$. Since the cost is negative, a movement through this path is cost-saving. Evaluate now how many units may we move through it. This is defined by the lowest shipment at a negative route cell of the path. This time the figure is again 11 (by pure coincidence). Multiplying it by the net cost of the path, which is $-5$, yields that we have saved 55 monetary units more (the reader is invited to verify it by himself).

Evaluating in this manner unfilled cells included in the paths of negative net cost and moving through these

Table 3.4-4

| Origin | Destination | | | | | Supply |
|---|---|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | |
| $A_1$ | 18 $\boxed{13}$ | 12 $\boxed{7}$ $\ominus$ | $\boxed{14}$ | $\boxed{7}$ | $\boxed{5}$ $\oplus$ | 30 |
| $A_2$ | $\boxed{11}$ | 15 $\boxed{8}$ $\oplus$ | 22 $\boxed{12}$ | 11 $\boxed{6}$ $\ominus$ | $\boxed{8}$ | 48 |
| $A_3$ | 6 | $\boxed{10}$ | 20 $\boxed{10}$ | $\boxed{8}$ | $\boxed{11}$ | 20 |
| $A_4$ | 14 | 8 | $\boxed{10}$ | 4 $\boxed{10}$ $\oplus$ | 26 $\boxed{15}$ $\ominus$ | 30 |
| Required | 18 | 27 | 42 | 15 | 26 | 128 |

paths as many units as possible, we will lower the total transportation costs even more. However, we may not do this without end as we may not diminish it beyond zero, hence sooner or later we would arrive at an optimal schedule. This schedule will no longer contain an unfilled cell with a negative net cost of a closed path. This feature will signal us that an optimal solution is found.

Linear programming has the solution methods that automatically evaluate unfilled cells of negative net cost, but we shall not pause on them as well as on other approaches to the transportation problem, which the reader can find in special texts, e.g. [5, 7, 8].

In conclusion, we say a few words on the situation when the sum of all demands is not equal to the available supplies, i.e.

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j \tag{3.4-7}$$

When $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$ (the supplies are in excess of the requirements) all the demands can be satisfied, but not all the supplies will be exhausted. This situation may be handled by introducing a fictitious destination, $B_f$, to absorb the difference (slack)

$$b_f = \sum_{j=1}^{m} a_i - \sum_{j=1}^{n} b_j \tag{3.4-8}$$

What should then be the costs of shipment from origins $A_i$ to the fictitious location? Naturally, they should be assigned zero values since in reality nothing will be shipped to $B_f$. Hence, for all origins $c_{if} = 0$.

We augment the transportation table with a dummy column corresponding to destination $B_f$ and containing zero transportation costs. Solution is then attempted in the usual manner to result in an optimal transportation schedule

$$\begin{array}{cccccc}
x_{11} & x_{12} & \dots & x_{1n} & x_{1f} \\
x_{21} & x_{22} & \dots & x_{2n} & x_{2f} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
x_{m1} & x_{m2} & \dots & x_{mn} & x_{mf}
\end{array}$$

We should keep in mind, however, that the slacks $x_{if}$ are shipped nowhere: they remain at the respective origins $A_i$.

The opposite situation with $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$ (not enough stocks available to meet all the demands) may also occur. This difficulty may be alleviated by cutting the required amount in some way. If it is insignificant how rightly the supplies are allocated, but the sole objective is to ship them at a least possible cost (no matter to what destinations they will be assigned), then we may introduce a dummy origin to supply the difference $\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$. We are not going to treat the topic in more detail, referring the reader to [6].

## 3.5 Integer Programming. The Concepts of Nonlinear Programming

A remarkable number of practical problems which have similar formulations to LP problems differ from them in that the sought values of the variables are stipulated to be integers (whole numbers). These problems are referred to as *integer programming (IP) problems.* The integer requirement substantially complicates their solution.

Consider an illustration of such a problem. Suppose that we are to evacuate $n$ precious items of art from a dangerous area. Their costs $c_i$ together with their weights $q_i$ are known. The number of items which we could evacuate is limited by the capacity of the available transportation facilities, $Q$. It is required to set up a most precious (highest priced) choice of the collection such as could be carried away in $Q$.

For the sake of consideration we introduce the so-called 0 - 1 integer variables $x_i$ ($i = 1, 2, \ldots, n$) such that $x_i = 1$ if we select the $i$th item and $x_i = 0$ if not. Then, for a selected lot, i.e. with assigned values

to $x_i$, the weight will be $q_1x_1 + q_2x_2 + \ldots + q_nx_n$. The restriction imposed by the carrying capacity is

$$q_1x_1 + q_2x_2 + \ldots + q_nx_n \leqslant Q \qquad (3.5\text{-}1)$$

The total value of the items, which we wish to maximize, is

$$L = \sum_{i=1}^{n} c_ix_i \Rightarrow \max \qquad (3.5\text{-}2)$$

On the surface this problem formulation almost does not differ from that of a common LP problem: find nonnegative values of variables $x_1, x_2, \ldots, x_n$ such that maximize the linear function of these variables (3.5-2) subject to the constraint (3.5-1). At first sight, it might seem to be handled in the same way as a LP problem, by solely augmenting it with the constraints accounting for the solitary property of each item:

$$x_1 \leqslant 1, \quad x_2 \leqslant 1, \ldots, \quad x_n \leqslant 1$$

Yet this addition does not go. A solution obtained in this way might appear fractional rather than integer and hence infeasible (can't we load a half of a sculpture or a third of a painting!). The considered problem is other than a LP, it is an integer program.

A second thought that comes to mind is whether the problem could be approached by solving its linear program part and then rounding off the obtained values either up or down to the next integer 0 or 1. Unfortunately, this approach is also invalid. A solution so received might even not meet the constraint (3.5-1), that is, it might violate the imposed weight limitation. Even if it obeys the restriction, it might lie far from an optimum. Some problems, though, admit this rounding approach. For example, if the production

scheduling problem of Section 3.1 would have a solution: 185.3 machines of first type must be assigned to production of first grade cloth, then we would simply round the figure to 185. If the point is to assign resources having the property of integrity (especially valuable or unique), the problem must be stated as an integer program.

The integer programming problems in general require a considerably more sophisticated approach for their solution than correspondingly sized conventional LP problems. Several solution techniques are known to be implemented for such problems; all of them, at whatever significant number of variables, are very complicated and laborious. We are not intended to cover these techniques here and refer those interested to the texts where more space is devoted to the topic (for example, [7, 8]).

The nonlinear character of the integer requirement is subtle enough to accommodate an unexpected variety of other nonlinearities. A few words therefore would be in order on nonlinear programming.

A general formulation of the nonlinear programming problem is as follows. Find nonnegative values of variables $x_1, x_2, \ldots, x_n$ such that maximize the nonlinear objective function of these variables

$$W = W(x_1, x_2, \ldots, x_n) \Rightarrow \max \qquad (3.5\text{-}3)$$

subject to the constraints of arbitrary type, say,

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &\geqslant 0 \\ f_2(x_1, x_2, \ldots, x_n) &\geqslant 0 \\ &\cdots\cdots\cdots \\ f_m(x_1, x_2, \ldots, x_n) &\geqslant 0 \end{aligned} \qquad (3.5\text{-}4)$$

No universally powerful general algorithm to solve nonlinear programming problems is known so far; each particular problem is approached depending on the form of nonlinearity of the objective function and the type of imposed constraints.

Nonlinear problems are frequent occasions in applicational arenas; they take place, for instance, where the costs vary not in direct proportion with the amount of acquired or produced commodities (say because of discount rising with a purchased lot). Many of them can be approximated by linear models (linearized), at least in an area close to an optimal solution. If the linearization cannot be succeeded, still problems of applicational value are normally pursued towards models with the nonlinearity that can be safely handled by some known algorithms. A particular recurrent type involves quadratic programming problems whose objective function is a second degree polynomial in variables $x_1, x_2, \ldots, x_n$, and the constraints (3.5-4) are linear inequalities (see [7, 8]).

A whole family of nonlinear programming problems can be solved to advantage by *penalty function algorithms* which reduce the constrained source problem to a sequence of "unconstrained" relaxed models the solutions of which approach a solution of the original problem. This is done by combining the constraints with the objective function in such a way that minimizing the combined 'penalty function' penalizes constraint violation. The basic idea may be exemplified as follows. Instead of imposing a strict constraint of the form $f(x_1, x_2, \ldots, x_n) \geqslant 0$, it may be added to the objective function $W(x_1, x_2, \ldots, x_n)$ in a form of penalty, say $af(x_1, x_2, \ldots, x_n)$, which would tend to prevent violation of the original constraint; here $a$ is the penalty coefficient (negative when the objective

function is maximized, and positive when minimized). Varying $a$ in absolute value one may gauge what variation it entails for an optimal solution $(x_1^*, x_2^*, \ldots, x_3^*)$ to hold one when the variations practically cease.

Another method which has proved of value in approaching nonlinear programs is *random search technique*, which, as the name implies, is the random (with a play of chance) investigation of the solution space for the optimum. This method as well as the mentioned above and others the reader can find (if needed) in [7-9].

Last but not least, a short mention is due for stochastic programming problems[7]. They arise in applications where optimal solutions have to be sought in the conditions of incomplete certainty. For example, some variables in the objective function and the constraints vary at random, i.e. can be realized only as probability statements. This programming is also referred to as chance-constrained programming. Some of these problems can be converted to ordinary deterministic models. This is the case, for example, when optimization is effected in the mean, the objective function linearly depends on the decision variables, and the only elements of randomness are coefficients of variables in the objective function. With this formulation, the solution can be optimized having substituted for the coefficients their mathematical expectations (the mathematical expectation of a linear function, as will be recalled, is equal to the linear function of the expectations of the arguments). More involved, however, are the situations where the imposed constraints

[7] Stochastic programming is a former misleading name of dynamic programming, too.— *Translator's note.*

are random functions. These and other problems of chance-constrained programming are extensively covered in [31].

In contrast to linear programming whose solution methods are well developed, settled, and never pose essential difficulties, the integer, nonlinear, and chance-constrained programming problems fall within the province of hard computational problems. In tackling them a resort is often made to approximate, heuristic optimization techniques, since analytical methods appear too laborious to handle even on a large computer. Not infrequently the efforts applied to gain an optimal solution entirely offset the reward that could be earned from this solution so that the game proves not worth the candle. This points out again to the necessity of applying a systematic approach to OR problems, which would account for not only an immediate winning in the operation at hand, but also the costs of pertinent optimization.

The soothing fact is that ways are still being devised of improving existing algorithms. This, coupled with the increasing efficiency of package programs as well as computer hardware, is constantly making it economically possible to solve larger models. On the other hand, the size of optimization models parallels the trend in computational capabilities. One should also not overlook that the decision made in choosing an optimization algorithm is an important stage, at times even the most critical.

# Chapter 4

# DYNAMIC PROGRAMMING

## 4.1 Concepts of Dynamic Programming

Dynamic programming (DP) is an optimization technique suitably adapted to work with the processes requiring sequential decision making to search for an optimum. These are known as *sequential decision processes.* A sequential decision process is loosely defined as an activity entailing a sequence of decisions taken toward some common goal.

To illustrate, imagine a process that breaks down into a series of sequential *steps* or *stages*, for example, activity of an industry for several fiscal years, or penetration by a group of airplanes several echelons of antiaircraft defence, or a sequence of tests established to control the quality of items of work. Some processes, or operations as we agreed to call them, break into stages in a natural way; in others the partitioning (for the sake of DP analysis) is performed artificially as, say, in aiming a missile which can be conditionally broken down into stages each of which takes some time to be accomplished.

Assume further that the operation in question consists of $m$ stages. Let the effectiveness of the operation be characterized by a performance measure, $W$, which will be succinctly referred to in this chapter as *reward*, or *return*. Assume that the ***total return*** of operation,

$W$, makes up of the returns at separate stages, $w_i$,

$$W = \sum_{i=1}^{m} w_i \tag{4.1-1}$$

i.e., has the property of additivity.

The concerned operation is a controlled process which means that we can choose the parameters affecting its performance and final result. Since it is a sequential decision process, at each stage we may select a *decision* upon which the *immediate result* depends at this stage and the *total return of the process*. The set of all sequential decisions constitutes a *policy pursued* on the process as a whole. Let $\mathbf{x}$ represent a policy, and $x_i$ an *immediate decision* at stage $i$, then

$$\mathbf{x} = [x_1, x_2, \ldots, x_m] \tag{4.1-2}$$

It should be kept in mind that the decisions are not necessarily numerals, they may be vectors, functions, or other mathematical forms.

The goal of DP is to establish a *sequence of decisions*, $\mathbf{x}$, such that maximizes the total return $W$:

$$W = \sum_{i=1}^{m} w_i \Rightarrow \max \tag{4.1-3}$$

The sequence of decisions $\mathbf{x}^*$ which yields the maximum gain will be called the *optimal policy* (or the optimal control, if one wishes it in terms of control). The sequence consists of optimal decisions made each at its stage:

$$\mathbf{x}^* = [x_1^*, x_2^*, \ldots, x_m^*] \tag{4.1-4}$$

The maximal reward which is earned with this policy will be referred to as $W^*$, that is,

$$W^* = \max [W(\mathbf{x})] \tag{4.1-5}$$

which is read as: the value $W^*$ is maximal of all $W(\mathbf{x})$ taken over all possible policies $\mathbf{x}$. A constraint imposed on the policies, that is, the fact that they must belong to a certain set $X$, is occasionally mentioned in the formula, namely,

$$W^* = \max_{\mathbf{x} \in X} [W(\mathbf{x})] \tag{4.1-5'}$$

Consider several examples of sequential decision processes to display what is meant in each of them under the term 'policy' and what is the return (measure of effectiveness) $W$.

EXAMPLE 1. A financial activity of a group of plants is planned for an $m$-year planning horizon. The company that controls the group supposes to invest in its development some monetary fund (resources) which must be allocated among the plants. As the time elapses, the resources invested into a plant are known to be partly expended (depreciated) and partly remained available for further reinvestment. Each plant makes a profit depending upon the amount of invested capital. At the beginning of each fiscal year all the available resources are reallocated among the plants. What amount of capital must be assigned to each plant at the start of a year to realize a maximum profit at the end of the planning horizon?

The total return (total profit) sums up of immediate profits over each year:

$$W = \sum_{i=1}^{m} w_i \tag{4.1-6}$$

and, hence, exhibits the property of additivity.

The decision $x_i$ made at the $i$th stage is to assign at the beginning of the $i$th year some amounts of the capital $x_{i1}, x_{i2}, \ldots, x_{ik}$ (the second subscript refers

to a particular plant) to the plants. Thus, in this problem a decision is a $k$-component vector:

$$\hat{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ik}) \tag{4.1-7}$$

Each decision enters into a sequence $\mathbf{x}$ which is a policy for the entire operation

$$\mathbf{x} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m] \tag{4.1-8}$$

The problem is to find an assignment of the capital to the plants and over years such that will maximize the total profit $W$.

While in this example the decisions are vectors, in the subsequent examples they manage with numbers.

EXAMPLE 2. A rocket has $m$ stages so that its orbiting may be broken down into $m$ steps at the end of which a stage burns out and separates. All the propelling stages together (i.e. without the spacecraft) have some stipulated weight

$$G = G_1 + G_2 + \ldots + G_m$$

where $G_i$ is the weight of the $i$th stage.

At the end of the $i$th leg of the escape trajectory (when the $i$th stage burns out and separates) a what-remains-of-the-rocket boosts its velocity by $\Delta_i$, the increment depends on the weight of the separated stage and the total weight of both the remaining stages and payload of the spacecraft. What weight must be allocated to each stage so that the terminal speed be maximal?

The measure of effectiveness to this problem will be the orbiting speed

$$V = \sum_{i=1}^{m} \Delta_i \tag{4.1-9}$$

where $\Delta_i$ is the return (speed increment) at the $i$th stage. A solution to the problem will be a set of weights of all stages

$$\mathbf{x} = [G_1, G_2, \ldots, G_m]$$

The optimal solution, obviously, will be the distribution of weights which maximizes the speed $V$.

EXAMPLE 3. A car owner prepares to use a car for $m$ years. At the turn of each next year he would face the following alternative situation:

(i) to sell the car and buy a new model,

(ii) to have it repaired and use it further,

(iii) or to use it without any repair.

The decision now is a choice of one of the three alternatives. They do not relate to figures, but the first may be assigned the value of 1, the second 2, and the third 3. What decisions must be made each year (i.e. how the assignments 1, 2, and 3 must be alternated) to minimize the total expenses for use, repair, and buying a new car?

The measure of effectiveness this time is pay-off rather than gain, but it does not matter. It may be expressed as

$$W = \sum_{i=1}^{m} w_i \tag{4.1-10}$$

where $w_i$ is the expenses for the $i$th year. The quantity $W$ must obviously be minimized.

A solution (policy) to this problem is a combination of the three numbers 1, 2, and 3, e.g.,

$$\mathbf{x} = [3, 3, 2, 2, 2, 1, 3, \ldots]$$

which is decoded as: the first two years use the car without repair, next three years it must be submitted

to repair, at the turn of the sixth year sell it and buy a new one, then again use it without repair, and so on. Any policy therefore is a collection of numerals

$$\mathbf{x} = [j_1, j_2, \ldots, j_m] \tag{4.1-11}$$

where each of them, $j_i$, is either of three numbers 1, 2, or 3. The problem is to choose a set (4.1-11) for which the quantity (4.1-10) is kept at a minimum.

EXAMPLE 4. A stretch of railway is being constructed between two locations $A$ and $B$ (Fig. 4.1-1).

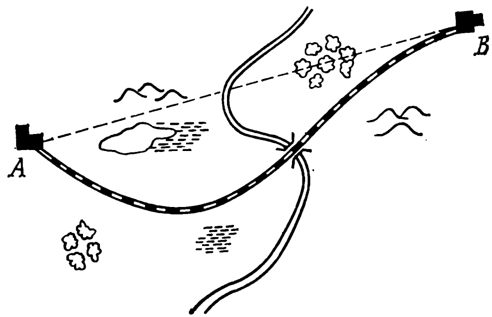


**Fig. 4.1-1**

It is to cross a rugged terrain ample with forest areas, hills, swamps, and a river calling for an inevitable bridge. The route must be chosen such that the total construction costs are a minimum.

This problem in contrast to the previous has no natural partition into steps. It can, however, be effected by breaking down the straight line section $AB$ into $m$ segments and drawing through the points of partition the lines perpendicular to $AB$. A step will

then be visualized as a transition from one such line to the next. If the lines are many, i.e. drawn substantially close to one another, then the segment of route at each step may be deemed approximately linear. A decision at the $i$th step materializes as an angle $\varphi_i$ which the railway makes at this point with line $AB$. A solution to the problem consists of a sequence of all decisions

$$\mathbf{x} = [\varphi_1, \varphi_2, \ldots, \varphi_m]$$

It is required to choose an optimal sequence $\mathbf{x}^*$ such that minimizes the total costs of construction

$$W = \sum_{i=1}^{m} w_i \Rightarrow \min \qquad (4.1\text{-}12)$$

Now that we have considered several examples of sequential decision processes, we may pause to discuss how these problems can be solved.

Any sequential decision problem may be approached in two ways. We may either seek for all decision variables at all steps at once, or build an optimal policy step by step, optimizing only one decision at each computational stage. Normally the second approach appears simpler than the first, especially when the number of steps is great.

This idea of sequential, step by step optimization underlies the method of dynamic programming. One stage is, as a rule, simpler to optimize than the entire process: it proves more efficient to optimize many times an easy to solve problem than once attack a complicated hard nut.

At first sight the idea might seem fairly trivial. Nothing seems more simple than to decompose a problem into a set of small steps when it is hard to

optimize as a whole. Each such step will be an individual, small, and easy to optimize operation. All we need is to take a decision such that the effectiveness of this step is optimal, isn't it?

Not at all! Way off target! The basic concept of dynamic programming has never suggested that each stage be optimized separately of the others. To the contrary, each decision must be selected with account of all its sequences in the future. What is the use of choosing a decision which maximizes the return at this stage if it deprives us of a possibility of a good reward at the subsequent stages?

Consider, for example, a planning for a group of manufacturing plants of which ones manufacture some merchandise while the others produce the relevant manufacturing equipment. The objective of this operation is to manufacture for $m$ years of the planning horizon a maximum amount of the merchandise. Assume that we schedule a capital investment for the first year. Viewing under the narrow angle of the objective for this year, all the available capital must have been invested into manufacturing of the merchandise. Will, however, this decision be efficient from the viewpoint of success for the whole activity? Obviously not, because this decision is wasteful and shortsighted. With the future in view, we must have invested a part of the capital into machinery. This surely would diminish the produced volume of merchandise for the first year, but would create the production capacity for its increase in the years to come.

Another example to the point. Suppose that in the problem on routing a railway construction from $A$ to $B$ we had been tempted with the easiest starting direction and started constructing on this path. What is the use of the easiness and economy on the first

leg if the very next would run the way into a swamp? It all implies that

> planning a sequential decision process we need to choose a decision at each stage with due consideration for its consequences at the stages to come.

A decision at the $i$th stage is taken so as to maximize the total return *for all the subsequent stages plus this one*, rather than with the point to maximize the return at *this* stage only.

There are no rules without exception, however. Among all the stages there is one which can be adopted without consulting the future consequences. Which one is this? Obviously the last. This stage is the sole to have a decision invoking the largest reward for it alone.

Therefore dynamic programming works normally backwards, reeling up from the terminal step to the beginning. The first decision, thus, is made at the last stage. How can it be made, however, if we do not know what was the result of the previous, the last but one stage; that is without a knowledge on the conditions under which we have to decide on the last step?

This is the point where lies the essence of the procedure. Working of the last stage, one needs to make various suppositions concerning what was the result of the last but one, $(m - 1)$st step and then to devise a *conditional optimal decision* at the $m$th step ('conditional' refers to the fact that the decision is made under the condition that the previous step has resulted so and so).

Suppose we have done it and know for each of possible outcomes of the last but one stage its conditional optimal decision with the attendant optimal return at the $m$th stage. Then we can optimize the decision

of the previous ($m$ — 1)st stage. Again make all possible suppositions on the outcomes of the previous, this time ($m$ — 2)nd stage and on the se suppositions find a decision at the ($m$ — 1)st stage such that maximizes the return at the two last stages (of which the $m$th has been already maximized). In this way we find for each outcome at the ($m$ — 2)nd stage the respective conditional optimal decision at the ($m$ — 1)st stage and the attendant optimal return at the two last stages. If the same process is continued we can back up all the way to the beginning, finding optimal decisions as we go.

Assume that all conditional optimal decisions together with the attendant returns at all the subsequent stages, i.e. from the given stage to the end, are known. It means that we know what to do, how to decide at this decision point, and what will be the resultant return at the end whatever the state of the process at the beginning of this stage. Now we are in a position to discard the attribute 'conditional' building a simply optimal policy, $\mathbf{x}^*$, and evaluating an optimal total return, $W^*$, to substitute those derived earlier as 'conditional'.

To prove, assume that we know the state $S_0$ of the controlled process at the first stage. Then we can make an optimal decision $x_1^*$ at this decision point. It will change the state of the process to a new one, say $S^*_1$. In this state the process arrives at the second stage. This, hence, gives us the knowledge on the conditional optimal decision $x_2^*$ which translates the process to a stage $S_2^*$ at the second stage, and so on. As to the optimal total return $W^*$ for the entire operation, we know it already, since it has been its maximality that we have in view making the decision at the first stage.

Consequently, in optimizing a policy by dynamic programming, a sequential decision process is as if "run" twice. First moving backwards to find conditional optimal decisions and conditional optimal returns for the "tail" stages of the process. Then in normal order when all we need is to read the prepared recommendations and evaluate the unconditional optimal policy $\mathbf{x}^*$ as a sequence of optimal decisions $x_1^*$, $x_2^*$, . . ., $x_m^*$.

The first run, that of conditional optimization, is much more involved and time consuming of the two. The second run requires almost no additional computation.

The author does not flatter herself with the hope that the reader who has never had a chance to meet dynamic programming before might get a true understanding of its basic idea from the above presentation of the method. A more comprehensive grasp could hopefully be obtained in considering examples of value. Some of them will be discussed in the subsequent section.

## 4.2 Solving Dynamic Programming Examples

This section is devoted to consideration (and even complete solution) of several simple (purposely simplified) dynamic programming problems.

**A cost-efficient path problem.** Recall the railway constructing problem of the previous section to solve it completely under strictly (and purposely) simplified conditions. Resuming, we need a path connecting two locations $A$ and $B$ of which the latter is situated, say, northeast of the former. To simplify the model, assume that the tracing is broken down into a series of steps on each of which we may move either east- or northwards. Therefore any path from $A$ to $B$ is a broken line whose

sections are parallel and perpendicular to either of coordinate axes (see Fig. 4.2-1) The costs to complete each of such sections are known. How to route a path from $A$ to $B$ to keep the total costs at a minimum?

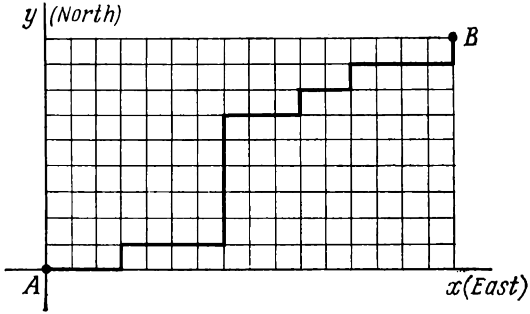


**Fig. 4.2-1**

We may adopt either of two ways: to compare by an exhaustive search all the possible alternatives and choose that of minimal costs (it would be a very difficult procedure when the number of sections is large), or to break down the process of moving from $A$ to $B$ into individual steps (one step equivalent to one line segment) and then optimize the path step by step. The second way turns out far more convenient. Here, as elsewhere in operations research there tells the superiority of direct search over naive exhaustive routine.

We illustrate the procedure by working it. Divide the distance from $A$ to $B$ in the eastern direction into 7 segments, say, and in the northern direction into 5 segments (basically, the division may be arbitrarily frequent). Then any path $AB$ will consist of $m = 7 + 5 = 12$ segments pointed either east or north.

Figure 4.2-2 displays the network, showing the cost of construction (in some conventional units) on each segment. It is required thus to select a path $AB$ for which the sum of segmental costs is minimal.

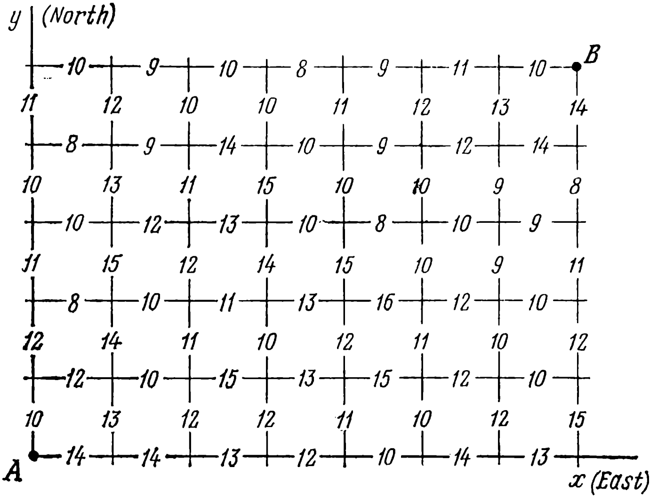


**Fig. 4.2-2**

The constructed path may also be visualized as a movement of a controlled system subject to issued decisions from the original state $A$ to the terminal state $B$. The state of the system before moving to a next leg will be described by two coordinates: the eastern ($x$) and the northern ($y$), both of which assuming only integer values ($0 \leqslant x \leqslant 7$, $0 \leqslant y \leqslant 5$). For each state of the system (a node in the network of Fig. 4.2-2), we are to derive a conditional optimal decision: either go north (decision N) or go east (decision E). The decision is made so that the cost of all remaining legs (including

the given step) is minimal. We will customarily refer to this cost as 'conditional optimal return' (although in this case it is a pay-off, not a gain) for a given state of the system before entering a next leg.

First perform conditional optimization moving backwards. Starting with the last, twelfth step, consider the upper right corner of the network, cut out in Fig. 4.2-3. Where could we be after the 11th step? Obviously, either in $B_1$ or $B_2$. If in $B_1$, then we have no choice and go east; it will cost 10 units. Put this cost into the node circle of $B_1$ and show the respective optimal decision by a short arrow pointing from the circle eastward. For node $B_2$, the decision is also enforced: go north, which entails a cost of 14 units encircled and arrowed at $B_2$. The conditional optimization of the last step is effected and the attendant conditional optimal return is evaluated and encircled at the respective node.

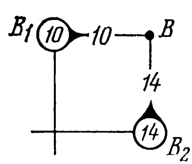


Fig. 4.2-3

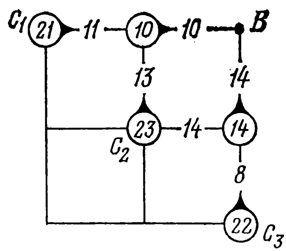


Fig. 4.2-4

Now we come to optimize the prior, 11th step. After the 10th step we could turn out at one of the nodes $C_1$, $C_2$, or $C_3$ shown in Fig. 4.2-4. Let us evaluate for each of them their conditional optimal decisions along with attendant returns. At node $C_1$ the decision is obvious:

go east with the cost to the end equal to 21 units. We home this figure into the circle at $C_1$. For $C_2$, the decision is no longer enforced: we may go both east and north. The first route entails a cost of 28 units summed up of 14 toward $B_2$ and another 14 to the end. If we go north, the route totals $13 + 10 = 23$ units. This implies that the conditional optimal decision at node $C_2$ is bound northward, which is marked by the arrow and the number 23 in the circle of node $C_2$. As to $C_3$, the decision is again enforced: go north (N) with a cost of 22 units.

Working backwards in a similar manner, we can find for each node its conditional optimal decision (N or E), put the respective return to the circle with an arrow pointing in the direction of departure. The return each time is evaluated on a recursive scheme; the return at a given step sums up with that already optimized and written in the circle to which points the arrow. Therefore, at each stage we optimize only the step under concern while the following are already optimized. The procedure results in the network presented in Fig. 4.2-5.

With the completed conditional optimization, we know where to go (an arrow) and what it will cost in total (figure in a circle) at whatever node we might turn out. The node circle at $A$ contains the optimal return for the whole path $AB$: $W^* = 118$ units.

All what is left now is to evaluate the unconditional optimal solution, that is, the path leading from $A$ to $B$ in the cheapest way. To do this we need only to go in the direction shown by the arrows. The optimal path is indicated in Fig. 4.2-5 by dual circles. The respective optimal policy is seen to be

$$\mathbf{x}^* = [\mathrm{N}, \mathrm{N}, \mathrm{N}, \mathrm{N}, \mathrm{E}, \mathrm{E}, \mathrm{N}, \mathrm{E}, \mathrm{E}, \mathrm{E}, \mathrm{E}, \mathrm{E}]$$

that is, the first four steps must be to the north, next two to the east, then one step to the north, and the five remaining to the east. This completes the solution.

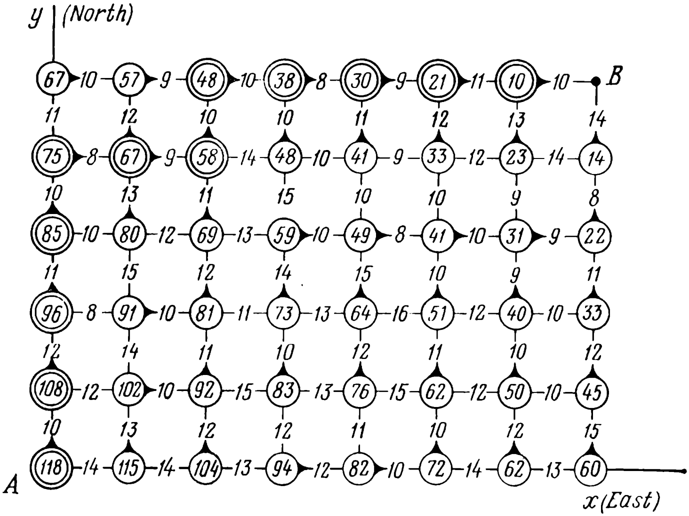


**Fig. 4.2-5**

Note that in a process of conditional optimization we may face a tie case when two or more decisions are optimal for a node, i.e. result in equal costs (expenditure of resources) from this node to the terminal node. In the above example, the node having coordinates (5, 1) enjoys two optimal decisions, N and E, resulting in a remaining cost of 62. We may choose any of them (we have preferred N but could equally have made for E). These cases of tie decision making are recurrent phenomena in dynamic programming. We shall not pause on them in the future, simply picking up one of the

equivalent options. This deliberateness can, obviously, affect an optimal policy, not the optimal return. Dynamic programming problems in general (as well as linear programs) have a unique solution not always.

Now return to our sample problem and try to solve it by a "naive" approach, choosing at each step, starting from the first one, the most cost-efficient (for this step) direction (if two, then any will go). This approach gives rise to the solution

$$\mathbf{x} = [\mathrm{N}, \mathrm{N}, \mathrm{E}, \mathrm{E}, \mathrm{E}, \mathrm{E}, \mathrm{N}, \mathrm{E}, \mathrm{E}, \mathrm{E}, \mathrm{N}, \mathrm{N}]$$

The costs for this path are $W = 10 + 12 + 8 + 10 + 11 + 13 + 15 + 8 + 10 + 9 + 8 + 14 = 128$, which is undoubtedly more than $W^* = 118$. The difference in this case is not very strong, though it might be far more substantial.

In the problem just solved the conditions were purposely submitted to a severe simplification. Certainly, it occurs to no one to trace a railway in steps, moving either strictly north or strictly east. This simplification was made solely to choose at each node out of two decisions only: N or E. We can, certainly, take more alternative directions and shorten the step; this will not make any principal difference, though, will make the computations more complicated and lengthy.

The problems similar to the discussed example can very often be met in applications, for example, in choosing the quickest path between two points, or most economical trajectory (in terms of fuel consumption) for an ascending aircraft aiming at specified altitude and speed.

Let us stop for a moment to discuss an interesting aspect. The observant reader might have noticed that in the above example problem, the origin and terminal points $A$ and $B$ do not principally differ from one

another. A conditional optimal solution might be deduced moving from the beginning to the end, rather than backwards, while the unconditional path on the back swings to the origin. It is indeed the case for any DP problem because the start and end may be interchanged. The resulting procedure will be entirely equivalent in the computational aspect to that described above, but somewhat less convenient to describe in words its ideas because it is easier to argue referring to the conditions which have already been created prior to the beginning of this stage than to those which are yet to come. In essence though, both approaches are equivalent.

**A resource allocation problem.** Dynamic programming proves very powerful technique in solving to advantage many economic applications (see, for example, [6, 10]). Consider a sample problem of this variety.

We (a board of directors) have at our disposal some capital funds (resource) which we are going to invest in, or allocate among, $m$ ventures. Each of the ventures is known to produce when invested with a capital $x$ a profit which is a function of the investment, $f_i(x)$. All the functions $f_i(x)$ ($i = 1, 2, \ldots, m$) are specified (they are certainly nondecreasing). How should we allocate the funds among the ventures to realize a maximum profit?

This problem can be readily handled by dynamic programming. Although the problem does not contain in its statement any time periods, the financial operation can be thought of as a sequence of transactions, assuming the first stage to be the investment in venture 1, the second stage the investment in venture 2, and so on.

We may again invoke the notion of a controlled system, S, which in this case may be visualized as the

funds or resources to be allocated. The state of the system **S** is characterized at each decision point by a single number $S$ representing the available, not yet invested, capital. The decisions in this problem are the amount of resources $x_1, x_2, \ldots, x_m$ allocated to the ventures. It is required to devise an optimal policy, that is such a sequence of decisions **x** which maximizes the total profit:

$$W = \sum_{i=1}^{m} f_i (x_i) \Rightarrow \max \qquad (4.2\text{-}1)$$

We shall approach the problem first generally to derive the needed formulas and then solve it with numbers. Find for each $i$th stage a conditional optimal return (from this stage through all the remaining stages to the end) if we have arrived at it having an available amount of resources $S$. Let $W_i(S)$ represent the conditional optimal return, and $x_i(S)$ the respective conditional optimal decision, i.e. the amount of resources invested in venture $i$.

Begin optimizing from the last, $m$th stage. Assume that we have arrived at this stage with a remained amount of funds $S$. What should we do? Obviously invest the entire funds $S$ in venture $m$. Therefore, the conditional optimal decision at the $m$th stage will be: invest in the last venture all the available resources $S$, i.e.,

$$x_m(S) = S$$

with the conditional optimal return

$$W_m(S) = f_m(S)$$

Letting $S$ assume a series of densely spaced values, we may compute $x_m(S)$ and $W_m(S)$ for each of them. The last stage may be deemed optimized.

Now go over to the previous, $(m - 1)$st stage. Suppose that we have come to this decision point with an available amount of resources $S$. Let $W_{m-1}(S)$ refer to a conditional optimal return at the two remaining stages $(m - 1)$st and $m$th (which has been already optimized). If we assign to the $(m - 1)$st venture a capital $x$, then the last venture receives the remaining $S - x$. The return at these two stages will be

$$f_{m-1}(x) + W_m(S - x)$$

and we must find an $x$ such that maximizes the return

$$W_{m-1}(S) = \max_{x \leqslant S} [f_{m-1}(x) + W_m(S - x)] \qquad (4.2\text{-}2)$$

Notice that we maximize over all possible values of $x$ which, however, cannot be larger than $S$ because we are unable to invest more than $S$. This maximum is the thought conditional optimal return at the last two stages, and the value of $x$ at which it is reached is the thought conditional optimal decision at the $(m - 1)$st stage.

Stepping further back we optimize $(m - 2)$nd, $(m - 3)$rd and earlier stages. In general, the conditional optimal return from all stages starting from a given $i$ to the end will be found with the formula

$$W_i(S) = \max_{x \leqslant S} [f_i(x) + W_{i+1}(S - x)] \qquad (4.2\text{-}3)$$

The respective conditional optimal decision $x_i(S)$ will be that value of $x$ at which this maximum is attained.

Proceeding in the same fashion, we shall arrive finally at venture 1. Here we have no need to vary $S$ since we know exactly the total amount of available

resources prior to entering the first stage, $R$, therefore

$$W^* = W_1(R) = \max_{x \leqslant R} [f_1(x) + W_2(R - x)] \quad (4.2\text{-}4)$$

Hence, the maximal return (profit) from all ventures is found. Now we need only to "read the recommendations". The value of $x$ at which (4.2-4) is a maximum relates to the optimal decision $x_1^*$ at the 1st stage. Having invested these funds in the 1st venture, we are to decide on the remaining $R - x_1^*$. "Reading" the recommendation for this value of $S$, we allocate to the second venture its optimal amount $x_2^* = x_2(R - - x_1^*)$ and so forth, to the end.

Consider now a numerical example. An input amount of resources valued $R = 10$ units must be optimally allocated to $m = 5$ enterprises. For simplicity assume that only integer amounts are invested. The functions of return are summarized in Table 4.2-1.

Table 4.2-1

| $x$ | $f_1(x)$ | $f_2(x)$ | $f_3(x)$ | $f_4(x)$ | $f_5(x)$ |
|---|---|---|---|---|---|
| 1 | 0.5 | 0.1 | 0.6 | 0.3 | 1.0 |
| 2 | 1.0 | 0.5 | 1.1 | 0.6 | 1.2 |
| 3 | 1.4 | 1.2 | 1.2 | 1.3 | 1.3 |
| 4 | 2.0 | 1.8 | 1.4 | 1.4 | 1.3 |
| 5 | 2.5 | 2.5 | 1.6 | 1.5 | 1.3 |
| 6 | 2.8 | 2.9 | 1.7 | 1.5 | 1.3 |
| 7 | 3.0 | 3.5 | 1.8 | 1.5 | 1.3 |
| 8 | 3.0 | 3.5 | 1.8 | 1.5 | 1.3 |

A common feature that might strike in all the columns is that starting with some level of investment the profit ceases increasing (in real life the phenomenon

corresponds to the fact that each enterprise is able to accommodate only a limited amount of investment).

Perform conditional optimization in the manner described above, starting with the last, 5th stage. Each time we come up to a next stage with an available resource $S$, we allocate at this stage an amount of resource, look up the return for this stage in Table 4.2-1, add it up to the already optimized return at all the subsequent stages to the end (considering that the remaining funds are lowered exactly by the allocated amount) and find the investment for which the sum maximizes. This investment relates to the conditional optimal decision at this stage and the maximum proper is the conditional optimal return.

Table 4.2-2

| $S$ | $i=5$ | | $i=4$ | | $i=3$ | | $i=2$ | | $i=1$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $x_5(S)$ | $W_5(S)$ | $x_4(S)$ | $W_4(S)$ | $x_3(S)$ | $W_3(S)$ | $x_2(S)$ | $W_2(S)$ | $x_1(S)$ | $W_1(S)$ |
| 1 | $\boxed{1}$ | 1.0 | $\boxed{0}$ | 1.0 | 0 | 1.0 | 0 | 1.0 | | |
| 2 | 2 | 1.2 | 1 | 1.3 | 1 | 1.6 | 0 | 1.6 | | |
| 3 | 3 | 1.3 | 2 | 1.6 | $\boxed{2}$ | 2.1 | 0 | 2.1 | | |
| 4 | 4 | 1.3 | 3 | 2.3 | 2 | 2.4 | 0 | 2.4 | | |
| 5 | 5 | 1.3 | 3 | 2.5 | 1 | 2.9 | 0 | 2.9 | | |
| 6 | 6 | 1.3 | 4 | 2.6 | 2 | 3.4 | 5 | 3.5 | | |
| 7 | 7 | 1.3 | 5 | 2.7 | 2 | 3.6 | 5 | 4.1 | | |
| 8 | 8 | 1.3 | 5 | 2.8 | 4 | 3.7 | $\boxed{5}$ | 4.6 | | |
| 9 | 9 | 1.3 | 6 | 2.8 | 5 | 3.9 | 7 | 5.1 | | |
| 10 | 10 | 1.3 | 7 | 2.8 | 5 | 4.1 | 7 | 5.6 | $\boxed{2}$ | $\boxed{5.6}$ |

Table 4.2-2 summarizes the results of conditional optimization over all stages. The table is seen to be

set up from the upper left corner to the right and then line by line downwards. The decision at the 5th stage is enforced: all the available funds are invested; the decisions at all other stages have to be optimized. The sequential backwards optimization from stage 5 to stage 1 yields a complete list of sequential decisions and the total optimal return $W^*$ for the whole operation, amounting to 5.6 in this case. The last two columns have only one row because the initial state of the system is known exactly: $S_0 = R = 10$. The optimal decisions at all stages are confined in frames. The optimal sequence of investments reads thus: allocate two units of 10 to enterprise *1*, five units to enterprise *2*, two units to enterprise *3*, none to enterprise *4*, and a unit to enterprise *5*. With this allocation the total profit attains a maximum of 5.6.

To provide the reader with an insight to how the table was set up, we perform a sample computation. Suppose, for example, we are to optimize a decision $x_3$ (7), that is, to decide on how many units we should allocate at stage *3*, where we have arrived with $S = 7$ units of resource available, and what maximum amount we can earn at all the remaining stages including the third. Assume all the stages following the third (i.e. 4th and 5th) have already been optimized and, hence, the columns for these two stages in the table are filled. Let us find $x_3$ (7) and $W_3$ (7). To effect this, compose an additional table (see Table 4.2-3). Its first column lists all possible investments $x$ at the third stage, which cannot be more than $S = 7$. The second column lists all what remains from $S = 7$ after this investment. The third column presents the return at stage *3* from the investment of amount $x$ in enterprise *3* (filled with reference to the column for $f_3$ $(x)$ in Table 4.2-1). The fourth column gives the optimal return at the remaining

(4th and 5th) stages on the condition that we have come up to stage *4* with what has been left of the resources (filled with reference to column $i = 4$ of

Table 4.2-3

| $x$ | $7-x$ | $f_3(x)$ | $W_4(7-x)$ | $f_3(x)+W_4(7-x)$ |
|---|---|---|---|---|
| 7 | 0 | 1.8 | 0 | 1.8 |
| 6 | 1 | 1.7 | 1.0 | 2.7 |
| 5 | 2 | 1.6 | 1.3 | 2.9 |
| 4 | 3 | 1.4 | 1.6 | 3.0 |
| 3 | 4 | 1.2 | 2.3 | 3.5 |
| $\boxed{2}$ | 5 | 1.1 | 2.5 | $\boxed{3.6}$ |
| 1 | 6 | 0.6 | 2.6 | 3.2 |
| 0 | 7 | 0 | 2.7 | 2.7 |

Table 4.2-2). The fifth column lists the sum of two returns: that attendant to the decision made at the stage and the optimized return from the following stages with the investment $x$ at the third stage.

Of all the returns in the last column, we pick up the maximal, $W_3(7) = 3.6$, which relates to the decision $x(7) = 2$.

A situation with a maximum (in an additional table of the type) attained at two or more $x$ rather than one can be easily handled by choosing any of them because this choice does not affect the reward. As we have already mentioned, the solution of dynamic programming problems need not in general be unique.

**A truck loading problem.** The approach of dynamic programming can successfully be used to tackle certain optimization problems discussed in Chapter 3, in particular some problems of integer programming. Notice

that the integrity requirement impeding the solution of LP problems, this time relaxes rather than complicates the procedure (as did the integrity of investments in the previous problem).

Consider by way of example a problem on loading a truck (a similar loading problem was treated in the previous chapter) which formulates as follows. A truck can carry a total of $Q$ tons. A certain collection of items (a sole thing of each) is available for shipment. Their weights and values are known (say, tabulated as in Table 4.2-4). Determine which of the items must be shipped to maximize the value of the loading which is $\leqslant Q$.

As can be readily observed, this problem in essence is very much the same as the previous (resource allocation among $n$ enterprises) but is somewhat simpler than that. To demonstrate, the process of loading can be thought of as consisting of $n$ stages; at each of them we decide whether or not to take a given item in the truck. The decision at stage $i$ is unity if we take the $i$th item, and zero if not. It means that at each stage we have only two decision alternatives and this is a very comfortable thought.

The state of the system prior to each next stage will obviously be described by the loading capacity, $S$, which still remains after some items have been loaded to the truck. For each value of $S$ we should find the total maximal value $W_i(S)$ of the items which still can be loaded on at a given value of $S$, setting each time $x_i(S) = 1$ if we take the item, and $x_i(S) = 0$ if not. Then we need to read these conditional recommendations, and the thing is done.

Now solve this problem in numerical form for six items whose weights and values are specified in Table 4.2-4.

Table 4.2-4

| Item | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Weight | 4 | 7 | 11 | 12 | 16 | 20 |
| Value | 7 | 10 | 15 | 20 | 27 | 34 |

The total loading capacity of the truck is $Q = 35$ weight units. It is required to point those items which must be included in the loading to maximize its value.[1]

As before, $S$ will be assigned only integer values. The conditional optimization of solution is indicated in Table 4.2-5, where each row gives for a respective stage (item) its conditional optimal decision $x_i$ (0 or 1) and conditional optimal return $W_i$ (the value of all the items remaining to be loaded at optimal decisions at all stages). We shall not dwell on how this table is prepared since the procedure parallels that of the previous problem, the only difference being 0-1 numbered decisions.

The bold-faced values in Table 4.2-5 are: the optimal return $W^* = 57$ and the optimal decisions providing for it, viz., $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 0$, $x_4^* = 1$, $x_5^* = 1$, $x_6^* = 0$. The solution thus is to load the truck with items *2*, *4* and *5* whose weight totals 35 (the exact coincidence of weight and loading capacity is not

[1] The items in Table 4.2-4 are ranged in the order of increasing weights; their values also increase from left to right which is natural but not necessary. It is intuitively obvious that loading by high-weight and low-value items would be unsound.

Table 4.2-5

| $S$ | $i=6$ | | $i=5$ | | $i=4$ | | $i=3$ | | $i=2$ | | $i=1$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x_i$ | $W_i$ | $x_i$ | $W_i$ | $x_i$ | $W_i$ | $x_i$ | $W_i$ | $x_i$ | $W_i$ | $x_i$ | $W_i$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 10 | | |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 10 | | |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 10 | | |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 10 | | |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 15 | 0 | 15 | | |
| 12 | 0 | 0 | 0 | 0 | 1 | 20 | 0 | 20 | 0 | 20 | | |
| 13 | 0 | 0 | 0 | 0 | 1 | 20 | 0 | 20 | 0 | 20 | | |
| 14 | 0 | 0 | 0 | 0 | 1 | 20 | 0 | 20 | 0 | 20 | | |
| 15 | 0 | 0 | 0 | 0 | 1 | 20 | 0 | 20 | 0 | 20 | | |
| 16 | 0 | 0 | 1 | 27 | 0 | 27 | 0 | 27 | 0 | 27 | | |
| 17 | 0 | 0 | 1 | 27 | 0 | 27 | 0 | 27 | 0 | 27 | | |
| 18 | 0 | 0 | 1 | 27 | 0 | 27 | 0 | 27 | 0 | 27 | | |
| 19 | 0 | 0 | 1 | 27 | 0 | 27 | 0 | 27 | 1 | 30 | | |
| 20 | 1 | 34 | 0 | 34 | 0 | 34 | 0 | 34 | 0 | 34 | | |
| 21 | 1 | 34 | 0 | 34 | 0 | 34 | 0 | 34 | 0 | 34 | | |
| 22 | 1 | 34 | 0 | 34 | 0 | 34 | 0 | 34 | 0 | 34 | | |
| 23 | 1 | 34 | 0 | 34 | 0 | 34 | 1 | 35 | 1 | 37 | | |
| 24 | 1 | 34 | 0 | 34 | 0 | 34 | 1 | 35 | 1 | 37 | | |
| 25 | 1 | 34 | 0 | 34 | 0 | 34 | 1 | 35 | 1 | 37 | | |
| 26 | 1 | 34 | 0 | 34 | 0 | 34 | 1 | 35 | 1 | 37 | | |
| 27 | 1 | 34 | 0 | 34 | 0 | 34 | 1 | 42 | 1 | 44 | | |
| 28 | 1 | 34 | 0 | 34 | 1 | 47 | 0 | 47 | 0 | 47 | | |
| 29 | 1 | 34 | 0 | 34 | 1 | 47 | 0 | 47 | 0 | 47 | | |
| 30 | 1 | 34 | 0 | 34 | 1 | 47 | 0 | 47 | 0 | 47 | | |
| 31 | 1 | 34 | 0 | 34 | 1 | 47 | 1 | 49 | 0 | 49 | | |
| 32 | 1 | 34 | 0 | 34 | 1 | 54 | 0 | 54 | 0 | 54 | | |
| 33 | 1 | 34 | 0 | 34 | 1 | 54 | 0 | 54 | 0 | 54 | | |
| 34 | 1 | 34 | 0 | 34 | 1 | 54 | 0 | 54 | 0 | 54 | | |
| 35 | 1 | 34 | 0 | 34 | 1 | 54 | 0 | 54 | 1 | 57 | 0 | 57 |

necessary; an optimal choice can afford some underloading).

In retrospect it is possible to observe that a simpler solution to this elementary example might have been to check all possible combinations of items for fitting in an exhaustive manner to choose finally that one whose value will be maximal. With a larger number of items, however, this would be a laborious way because the number of possible combinations increases rapidly. Dynamic programming cannot be hampered with any rise in stages: it entails only a proportional increase in amount of computations.

## 4.3 A General Form of DP Problem. The Principle of Optimality

The trivial DP problems considered above provide insight to the basic idea of the method: step by step optimization performed "conditionally" in one direction and "unconditionally" in the opposite. Dynamic programming is a very powerful technique to optimize a control; it cannot be impeded by either the integrity requirement, or nonlinearity of the objective function, or the type of imposed constraints. However, in contrast to linear programming, DP cannot be reduced to a single computational algorithm; it can be relegated into a computer only after respective formulas are derived which often is not a simple task.

This section will display a sort of a "review for a newcomer" discussing DP problem formulation, approach to solution, form of writing, etc.

The first question to answer in problem formulation is: what input parameters (sometimes called *state variables*) can the state of a controlled process (or system) be characterized with prior to each stage? A suc-

cessful choice of a set of these parameters often decides whether or not the optimization problem can be solved to advantage. In the three sample problems we solved in the previous section, the state of the process was described by a very small number of parameters: by two coordinates in the first example, and by one numeral in both the second and the third. These ultrasimple problems, however, are not frequent occasions in practice. More common are those where the state of the system is described by many state variables so that an exhaustion of all decision alternatives and evaluation of an optimal decision for each of them becomes a cumbersome procedure. It becomes even more so when the number of possible decision alternatives is large. Richard Bellman, who developed the concepts of partial optimization underlying dynamic programming, claimed these situations as being "cursed" by multiple dimension scourging not only dynamic programming, but also all the other optimization techniques. A common method of solving DP problems is on the computer rather than by hand; however, even modern large-capacity computers cannot match many such problems. It is very important therefore to formulate a DP problem correctly and economically without overloading the formulation with minor details, and simplifying as far as possible the description of controlled system (i.e. model) to shorten the number of decision alternatives. The reader might infer from what has been said that success in dynamic programming in many respects depends on the skill and expertise of the OR researcher.

The second task after having been completed with system description and list of decisions is to break it down into stages. Some fortunate situations provide it in the very formulation of the problem (say, fiscal

years in economic problems), but more often the partitioning has to be invented, as we have done it in the railway path problem of Section 4.2. Should we have not confined ourselves within the primitive model of only two decisions (N and E), it would have been more convenient to break the path into steps in another way, for example, assuming a step is a passing from one line section parallel to the ordinate axis to a next. Another partitioning approach would be to consider instead of straight line sections a set of concentric circles or other curves extending from the origin at $A$. All these approaches to shortest path selection would inavoidably restrict a choice of possible directions. Regarding the steps as transients from one constant abscissa line to another, it is easy to realize that the available manifold of decisions does not provide for back up, i.e. from an eastern to a western line. For a majority of applications such restrictions are natural (for example, a rocket launched off the earth and supposed to move through an efficient escape trajectory could hardly be imagined to tilt its engines and perform a "nip" down to the earth). One may meet, however, an entirely different situation. For instance, a path through a rough terrain (as in mountains) spirals appearing at some sequentially further points geometrically closer to a point of departure. DP problem formulation, in particular choosing a system of coordinates and a breaking-into-stages approach, should consider all reasonable constraints imposed on the decisions involved.

What policy should we adopt towards the number of stages $m$? At first glance it may seem the larger $m$ the better. This, however, is not entirely so. With increasing $m$ the amount of computations also increases, which is not always justified. The number of stages should

be selected with account of two conflicting circumstances: (i) the stage must be sufficiently short to enable a simple optimization procedure for the decisions to be made, and (ii) the stage must not be too short to avoid unnecessary computations which may only complicate the search for an optimal solution, rather than materially change the optimum found for the objective function. In any application, it is the "acceptable" solution, which may insignificantly differ from the optimal in the return $W^*$, that is of interest, not a strictly optimal solution.

We are now ready to state a general principle which is a basic one to the solution of all DP problems (occasionally also termed the principle of optimality).

> Whatever the state of a process concerned prior to a given stage in a sequence, a decision must be made such that maximizes the return at this stage summed with the optimal return at all the remaining stages.

The readers in command of usual courses of calculus and probability might comprehend this principle completely only if it follows a discussion of certain examples. That is the reason why the principle is presented not at the beginning of the chapter—which would be natural for a mathematician—but here, when the necessary examples have already been considered.

To summarize, we review the steps of DP problem formulation which might be of value to a novice in the field.

1. Select the parameters (state variables) which characterize the state of the controlled system at each decision point.
2. Break down the operation into stages.
3. Evaluate the set of decisions $x_i$ for each stage along with the imposed constraints.

4. Determine the return which will result from a decision $x_i$ at stage $i$, if prior to that the system was in a state $S$, i.e. to state the return function

$$w_i = f_i(S, x_i) \tag{4.3-1}$$

5. Determine how the state $S$ of the system will change in response to a decision $x_i$ made at stage $i$. The system will assume a new state

$$S' = \varphi_i(S, x_i) \tag{4.3-2}$$

The functions governing transitions (4.3-2) must also be written[2].

6. Put down the main recursive equation of DP (*functional equation*) yielding the conditional optimal return $W_i(S)$ (beginning with the $i$th stage to the end) in terms of the already known function $W_{i+1}(S)$

$$W_i(S) = \max_{x_i} [f_i(S, x_i) + W_{i+1}(\varphi_i(S, x_i))] \tag{4.3-3}$$

To this return there corresponds the conditional optimal decision $x_i(S)$ at stage $i$ (it should be emphasized that the changed state $S' = \varphi_i(S, x_i)$ must be substituted for $S$ in the already known function $W_{i+1}(S)$).

7. Perform conditional optimization of the last, $m$th stage, specifying a set of states $S$ one stage apart from the terminal state and computing for each of them the conditional optimal return by the formula

$$W_m(S) = \max_{x_m} [f_m(S, x_m)] \tag{4.3-4}$$

to find the conditional optimal decision $x_m(S)$ which maximizes this return.

---

[2] Notice that the arguments of (4.3-1), (4.3-2) are generally collections of numbers, i.e. vectors, rather than single numbers.

8. Perform conditional optimization for the $(m - 1)$st, $(m - 2)$nd and earlier stages with (4.3-3), setting there $i = (m - 1), (m - 2)$, and so on, and show for each of the stages the conditional optimal decision $x_i(S)$ maximizing the expression.

Note that if the state of the system in the initial moment is known (which is the case as a rule), then there is no need to vary the state of the system at the first stage: the optimal return for a given initial state $S_0$ can be found directly. The obtained result is the total optimal return for the entire operation

$$W^* = W_1(S_0)$$

9. Perform unconditional optimization of the solution thus obtained by "reading" the corresponding recommendations for each stage. With the optimal decision $x_1^* = x_1(S_0)$ made at the first stage change the state of the system by (4.3-2) to find the optimal decision at the second stage $x_2^*$; repeat the cycle at all the subsequent stages to the end.

* * *

Pause to make a few additional general comments. So far we have considered only additive DP problems, i.e. those where the total return results as the sum of returns from sequential stages. Dynamic programming can, however, be also implemented on the problems with a multiplicative criterion, i.e. expressed as a product

$$W = \prod_{i=1}^{m} w_i \tag{4.3-5}$$

(if only the rewards $w_i$ are all positive). The solution of these problems is identical to that of additive crite-

rion with the sole difference that (4.3-3) has a product sign $\times$ in place of the plus sign, viz.,

$$W_i(S) = \max_{x_i} [f_i(S, x_i) \times W_{i+1}(\varphi_i(S, x_i))] \quad (4.3\text{-}6)$$

To conclude the chapter, a few words are in order on the DP problems dealing with infinite sequential decision processes. In some applications, the operation has to be scheduled for an indefinitely long-time period and an optimal solution to the problem may be wished irrespective of the number of the stage at which the operation terminates. A convenient model for such problems is infinite sequential decision process in which all the stages are equal in that none of them is terminal. This process obviously calls for the return function $f_i$ and the state transition function $\varphi_i$ be independent of the number of stage. We refer the interested reader to classical work [10]. Generally, a more pronounced insight into dynamic programming can be gleaned from the texts [6, 10, 11, 7].

# Chapter 5

# MARKOV PROCESSES

## 5.1 The Concept of the Markov Process

So far we have considered mainly deterministic OR problems along with respective optimization methods. Henceforth to the very end of the book our concern will be OR problems under uncertainty. This chapter will be focused on a comparatively favorable case, that of stochastic uncertainty (see Section 2.2). The unknowns in such a problem are random variables (or functions) whose probability characteristics either are known or may be inferred from experiment. Our present consideration will be mainly devoted to what we agreed to call primal problems of OR, i.e. to model development for certain random phenomena, touching only shortly on the inverse problems, i.e. solution-optimization techniques, because as a rule they are sophisticated. For stochastic problems, even model development is a hard task, let alone optimization. The majority of the problems cannot enjoy a simple mathematical model such that would enable us to find explicitly (analytically) the wanted quantities (effectiveness measures) as functions of the operation's constraints and decision variables. In some special situations this model construction can, however, be succeeded. This is the case when the operation in question is (exactly or approximately) the Markov process or chain.

A definition of the Markov process is deferred for a while, to give way for a discussion on the stochastic process in general.

Let a certain physical system change its state in time (go from one state to another) in a random, unknown beforehand manner. Then we will say that a random (stochastic) process runs in the system.

The physical system may be materialized in a variety of ways, say, as a device, a group of such devices, an enterprise, an industry, a living organism, a population of species, and so forth. Randomness and uncertainty, though in different extent, are intrinsic to majority of actual systems.

Consider as an example of such system a spacecraft being put into an orbit. The orbiting will unavoidably suffer from random errors, that is deviations from the specified trajectory, necessitating a correction to be performed (if it were not for these random errors, no correction would be needed). Hence a putting into an orbit is a stochastic process.

Now dive closer to the earth, in the dense atmosphere. A physical system this time is an aircraft performing a flight at a specified altitude and to a certain route. Is this process a stochastic one? Certainly, yes, since the flight would experience the influence of atmospheric eddies and other fluctuations which may result in random disturbances and fluctuations in its performance and accuracy of the flight schedule (anyone who suffered from a rough air or a lengthy waiting at an airport due to delayed airtraffic would evidence to the point).

One more example, this time of a system realized as a facility involving several devices which used to fail from time to time, and when failed being replaced or recovered. A process running in this system is undoubt-

edly random. We may find a lot of random processes in the world around us. Moreover, an example of a nonrandom process is more hard to invent than that of a random process. Even if we turn to our watches—a classical example of precise operation coined the attribute 'clockwork'—the process is liable to random variations, whether lagging, running, or occasional stop.

Does it mean that all the natural processes are stochastic? Strictly speaking, it is so since random fluctuations are intrinsic to any process. Yet as long as these disturbances are insignificant and influence the parameters concerned only slightly, we may neglect them and treat the process as a deterministic, i.e. nonrandom, one. We have to account for random factors when they directly concern us. For example, in scheduling flights an airline company may discard random fluctuations of an airplane about its mass center, whereas a designer of an autopilot may not. A majority of processes with which we deal in physics or engineering are essentially random but only some of them we consider as random—when we are in pressing need of that.

Having substantially elucidated the notion of random process, now we come over to define the *Markov process.*

> A random process is referred to as Markov if for any moment of time $t_0$ its probability characteristics in the future depend only on its state at time $t_0$ and are independent of when and how this state was acquired.

This important definition is worth discussing in a little more detail. Let at a present moment of time $t_0$ (refer to Fig. 5.1-1) the system in question assume a certain state $S_0$. We view the process from the side

and know at $t_0$ the state of the system $S_0$ and the history of the process, i.e. all events happened at

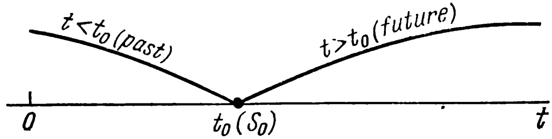


**Fig. 5.1-1**

$t < t_0$. We naturally wish to know what will be in the future, at $t > t_0$. Can we forecast it? An explicit forecasting is impossible because our process is random, hence unpredictable. Some probability characteristics of the process we, however, may evaluate. For instance, the probability that in a certain time the system will turn out in a state $S_1$ or will remain in $S_0$, or some other event.

The key point is that a *Markov process* (or *chain*) substantially facilitates this probability forecast as compared to the non-Markovian. Basically it is so because if a process at hand is Markov, then we may do the forecasting taking into consideration only the present state of the system, $S_0$, and putting aside the process history, i.e. its behavior at $t < t_0$. Or, putting it another way, the future in the Markov process depends on the past only in terms of the present.[1]

---

[1] The Markov processes where the probability of the next outcome depends only on the outcome immediately before are referred to as the first-order (or 'lag-1') Markov chains. If the outcome depends on other than the immediately prior result, it is a higher-order chain. For example, a second-order chain describes a process in which an outcome depends upon the two previous outcomes, etc.— *Translator's note.*

The Markov process may be exemplified by a Geiger-Müller counter which performs each next count when a nuclear particle enters the chamber. The state of the system's display is defined by the indication, i.e. the number of particles which struck the tube prior to that moment. Let the counter indicate $S_0$ at $t_0$. The probability that the count at $t > t_0$ will be $S_1$ particles (or not less than $S_1$) does depend on $S_0$, but is independent of the particular instants at which previous particles entered the tube.

Some recurrent applications may be modeled, if not exactly then to certain approximation, by the Markov chain. To illustrate, consider an air combat. The state of the battle is characterized by the number of aircraft remained in the battle (not shot down) $x$ for opponent I and $y$ for opponent II by a certain time. Assume that at $t_0$ we have the data on both parties, $x_0$ and $y_0$. We would like to know the probability that battling airplanes I will outbalance by a moment $t_0 + t_1$. What does this probability depend on? First of all, on the state which the system occupies at $t_0$; it almost does not depend on when and in what sequence airplanes were shot down prior to $t_0$.

Now that the Markov process became a more or less evident model we are going to expose it from an unexpected side: virtually any process may be modeled as Markovian if all the parameters of the "past" on which the "future" depends be included in the "present". Imagine, for example, an item in use in an industrial context, say, which is subject to failure and prompt restoration from time to time; at a time $t_0$ it is still healthy and we wish to know the probability that it will be sound for another period $t_1$. If the present state of the item is thought of simply as 'the item is sound' then the process is obviously non-Markovian because

the probability that it will not fail for another period $t_1$ depends generally on how long it has been already operating and when it was submitted to the last repair. If both these parameters (the total operation time and the time elapsed from the last failure) were included in the present state of the item, the process might have been deemed Markov. However, this boosting of the present at the expense of the past need not always be useful since (if the number of historic parameters is great) not infrequently it increases the size of the problem making it not amenable to available computational facilities. Therefore in what follows, speaking of the Markov chain we shall think it of as a simple, non-subtle process having a small number of parameters defining the "present".

Markov processes in a "pure" form are non-common in applicational areas, though, non-seldom are those for which the effect of "history" may be neglected, and hence Markovian modeling be used to advantage. In not a long while we will demonstrate how it can be done.

Stochastic processes may be loosely classed according to the type, *discrete* or *continuous*, their time variable and outcomes (states) belong to. Of all possible combinations of discrete (continuous) state/discrete (continuous) time processes we will be interested in the *discrete-state/continuous-time* process possessing the Markov property as having great significance in OR applications. We can imagine this process as that in which all the possible states can be indexed in advance and the moments of transitions from one state to the next are not fixed beforehand (as in the case of processes developed in discrete time), rather they are random so that a transition may occur virtually at any time and proceeds jumpwise, in no time.

This process may be exemplified by a facility consisting of two units each of which can randomly, i.e. at an arbitrary moment of time, fail, whereupon it is immediately submitted to repair (renewal) which also lasts a random period of time, unknown in advance. The possible states of the system can be indexed as follows:

$S_0$ = both units are healthy operating,
$S_1$ = first unit is being repaired, second healthy,
$S_2$ = second unit is being repaired, first healthy,
$S_3$ = both units are being repaired.

The transients from one state to another occur virtually in no time at random instants of failure or termination of a renewal period.

A convenient approach to analysis of discrete-state stochastic processes is by invoking the geometrical analogy in the form suggested by graph theory. The states in a *graph* are depicted as some geometric figures (rectangles, circles or points) and called *vertices*, and the possible transients are shown by lines, called *edges*, or by arrows indicating the direction of a transition, and then called *arcs*. We will display the states as rectangles placed in the vertices and containing symbols of states inside.

A graph prepared for the above example is shown in Fig. 5.1-2. The arc leading from $S_0$ to $S_1$ implies the transition which occurs when the first unit fails; the arc pointing backwards, from $S_1$ to $S_0$, signifies the transition at the moment of terminating the renewal of this unit. The other arcs can be assigned by similar argument.

The astute reader might ask why an arc which would join $S_0$ and $S_3$ directly is absent. Can it not be that both units fail simultaneously, say, because of short cir-

cuit? A legitimate question. An answer is that we suppose the units fail independently of one another

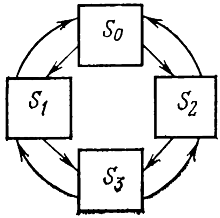


**Fig. 5.1-2**

and the probability that they fail simultaneously is believed to be negligible. (This statement will receive a more explicit substantiation later.)

If a process developing in the system with discrete states and continuous time is Markovian, then it can be described by a relatively simple model. However, prior to develop this model, we would find it useful to make familiar with the notion of 'arrivals' which is of significance for this and next chapters.

## 5.2 Arrivals Defined

A sequence of uniform events, called *customers*, which occur at random times, say, arrive at random at a *servicing facility*, is referred to as *arrivals*, or an *input* (*process*) to this facility. Examples may be arrivals of calls at a telephone exchange, arrivals of failures occurring within a computer, input of trains to a railway station, arrivals of nuclear particles at a Geiger-Müller tube, etc.

Arrivals may be visualized graphically as a series of points on the time axis $Ot$, shown in Fig. 5.2-1; one should not forget only that the position of each point is

random and Fig. 5.2-1 depicts a single realization of a process.

Speaking of arrivals as a series of events that occur at random, one should keep in mind that the events,

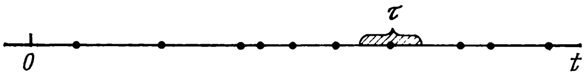


**Fig. 5.2-1**

realized say as a flow of customers, have no inherent probabilities. The probabilities are now in a possession of other, derivative events such as: "at time interval $\tau$ exactly two arrivals occur" (see Fig. 5.2-1), or "at least one customer arrives in the interval $\Delta t$", or "the inter-arrival interval will be equal or more than $t$".

An important characteristic of arrivals is *intensity* (or *arrival rate*), $\lambda$, stated as the mean rate of arrival per unit of time. The intensity may be constant or variable, say dependent on time $t$. For example, a stream of cars through a road crossing has a higher intensity at day time than at night, and is more intense in rush hours than in other hours of a day time.

An input is called *regular* or that of *constant arrivals* (or *intervals*), if customers arrive in regular, equally spaced intervals. More common in applications are, however, arrival patterns whose *inter-arrival intervals* vary at random.

An arrival pattern is termed *stationary* if its probability characteristics are independent of time. The intensity of this pattern need be constant. It does not mean at all that the number of customers arriving per unit time must be constant. The arrivals (if only they are not from a regular source) necessarily occur denser or rarer as, for example, is shown in Fig. 5.2.-1.

Important is that for a stationary input these density variations do not occur on a regular basis, and although one unit interval may enjoy more customers than another, the average number of customers per unit time is constant and hence independent of time.

A typical mistake which a novice is liable to commit is to take random clusters and rarefactions of arrivals for variations in arrival rate. We ought to caution the reader on this point.

As a rule, the deviations of a stationary behavior can be referred to some physical causes. It is natural, for example, that the arrival rate of calls arriving at a telephone exchange at night is lower than at day time (people used to sleep at night). More dense flow of customers arriving at a shop after working hours can also be explained. If an arrival pattern has a striking tendency to condense and rarefy (especially periodically), then a physical cause might be suspected and its evaluation should be attempted.

Arrivals of a stationary pattern (at least on a limited time interval) commonly arise in applications. To illustrate, arrivals of calls at a telephone exchange between 1 and 2 p.m. are virtually stationary; similar arrivals over the entire day would already be non-stationary.[2]

We have what is called an *independent* input process if for any two non-overlapping time intervals $\tau_1$ and $\tau_2$ (see Fig. 5.2-2) the number of arrivals that occur in one of them is independent of that in the other. It implies essentially that the arrivals occur independently of

[2] Observe that the situation is the same with most processes which we used to refer to as stationary in physics and engineering; actually they are stationary only over a restricted time interval and its extension to infinity is but a convenient technique.

each other, due to its own cause each. For example, a stream of passengers entering a railway station is

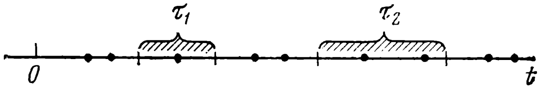


**Fig. 5.2-2**

virtually independent. Whereas a flow of customers leaving a counter in a shop cannot be deemed independent (at least because an interval of departure between any two of the customers cannot be shorter than a minimal time required to serve any one of them). It is similar with trains arriving at a station: there always exists a minimal time interval between each pair of them, established on safety grounds. Incidentally, if the minimal interval between arrivals is much less than the mean inter-arrival interval, $\bar{t} = 1/\lambda$, the effect of incomplete independence may be neglected.

An input process is referred to as *single arrivals* if the customers arrive one by one, not in batches. To illustrate, a stream of customers arriving at a barber's shop or queueing at a doctor's office is normally of a single variety, which cannot be said of wedding parties arriving at a registry office in a large city. Arrivals of trains at a station are single, whereas arrivals of the cars are not. For single arrivals input, the probability of two or more customers to arrive within a small time interval $\Delta t$ may be neglected.

We will call an input *Poisson* if it possesses the three following properties at once: it is stationary, single, and general independent (i.e. is *completely random*). It is the simplest arrival pattern mathematically and the most commonly useful one in applications. By the way, the simplest arrival physically—the regular or

constant-intervals input—is not so simple to deal with mathematically since it lacks general independence: its arrival instants are related by a strong functional dependence. Such an input process cannot be created

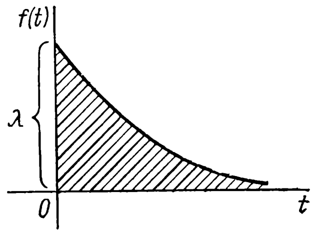


Fig. 5.2-3

without special efforts to sustain its regular intervals of arrival.

The role the Poisson arrival process plays among other arrival patterns is somewhat akin to that the Gaussian (normal) distribution does for the family of distribution models. A superposition of a sufficiently large number of mutually independent, stationary, and single arrivals (of comparable intensity) yields an arrival pattern close to the Poisson scheme.

It can be readily demonstrated (consult any text on probability, say [12]) that a Poisson input of arrival rate $\lambda$ has inter-arrival interval distributed exponentially with the density

$$f(t) = \lambda \exp(-\lambda t) \quad (t > 0) \tag{5.2-1}$$

shown in Fig. 5.2-3; here $\lambda$ is the parameter of the exponential distribution. For exponentially distributed random intervals, the mathematical expectation, $m$, is the quantity inversely proportional to the para-

meter, and the standard deviation is equal to the mean:

$$m = \sigma = 1/\lambda \tag{5.2-2}$$

As a 'measure of randomness' for a nonnegative random variable, probability theory often exploits the coefficient of variation:

$$v = \sigma/m \tag{5.2-3}$$

From (5.2-2) and (5.2-3) it follows that for an exponentially distributed variable, $v = 1$, i.e. for the arrivals distributed in the Poisson process the coefficient of variation for intervals between arrivals is unity. Obviously, for a constant intervals input process whose inter-arrival interval is not random at all ($\sigma = 0$), the coefficient of variation is zero. Most processes met in applicational areas have the coefficient of variation ranging between zero and unity, hence the coefficient may be employed as a measure of regularity, i.e. as a measure of deviation from the Poisson process. The closer it is to zero, the more regular is the input process. The Poisson process, being completely random, is the least regular of all applicational inputs.[3]

A convenient approach to arrival process computations is in using what we shall call an 'elementary probability'. We will define it with reference to a Poisson input of arrival rate $\lambda$ and a small time interval $\Delta t$. An elementary probability is such that predicts at least one arrival at this interval. It can be readily shown (we will not go into that at length) that the elementary probability is

$$p_{\Delta t} = \lambda \Delta t + o\,(\Delta t) \tag{5.2-4}$$

[3] One might invent a process for which $v > 1$, but nature almost never stages a state like that.

where the symbol $o(\Delta t)$ denotes a quantity that becomes negligible compared with $\Delta t$, as $\Delta t \to 0$. In other words, for a Poisson input, an elementary probability is equal to the arrival rate multiplied by the length of an elementary interval. Note that owing to independence, an elementary probability does not depend on whatever arrivals and when occurred earlier.

Observe also that accurate to higher-order infinitesimals, the probability of at least one arrival in $\Delta t$ is equal to the probability of exactly one arrival in this interval. It follows from the fact that the Poisson arrival process is single.[4]

We say that an input process is *general independent*, or *recurrent*, if it is stationary, single, and the intervals between arrivals, shown as $T_1, T_2, T_3, \ldots$ in Fig. 5.2-4, are independent and distributed with a general distribution function, say of a density as that in Fig. 5.2-5. Consider an example of an input process having a general independent form. A device, say a tube, functions continuously until it fails. A broken device is promptly replaced by a new one. If each subsequent device fails independently of the others, then arrivals of failures will be of a general independent form.

Another example of this input process. A retailer in a shop continuously serves customers (as is the case in maximum load hours). Serving a customer takes a random period of time. The intervals between served customers will be recurrent if service times of the customers are independent, that is, if a friendly chat between the retailer and a customer cannot influence service times of others.

[4] Recall the discussion on absent arcs from $S_0$ to $S_3$ and back in the graph of Fig. 5.1-2. It may be explained by the assumption of both arrivals of failures and arrivals of renewals being single.

Obviously, the Poisson input is a particular case of the general independent process when the intervals between arrivals are distributed exponentially as in

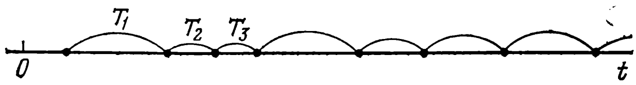


**Fig. 5.2-4**

(5.2-1). Another degenerate case of the general independent input is the constant arrival process, whose constant intervals are not random at all.

A whole family of recurrent arrival processes ordered to various degrees can be obtained by "screening" the

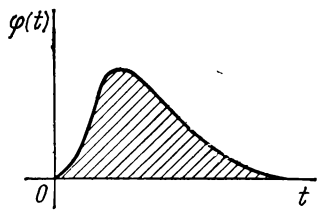


**Fig. 5.2-5**

Poisson input process. Let, for example, a stream of customers to an office be Poisson. Arriving customers are assigned at the reception board to clerks (servers) in strict rotation, i.e. the first customer to the first desk, the second to the second desk, and so on. If the office has $n$ servers, then each of them would have to handle $n$-Erlangian arrivals of customers. The input process like that realizes from a Poisson process if we keep each $n$th arrival and screen off all the rest. The

Poisson input is but a first-order Erlangian arrival process. It may be demonstrated that with this screening of a Poisson arrival process the respective coefficient of variation for inter-arrival intervals reduces; and when the order $n$ increases, the Erlangian input approximates to a regular process. The coefficient of variation for inter-arrival intervals of a $n$-Erlangian input is $v^{(n)} = 1/\sqrt{n}$. The special Erlangian inputs form a family of arrivals variously ordered: from complete disorder (Poisson) to complete order (regular arrival process).

## 5.3 The Kolmogorov Balance of State Equations

In dealing with discrete-state/continuous-time Markov chains, it would be convenient to think of the system's transitions from one state to another as occurring under the influence of some arrivals (incoming calls, arrivals of failures, arrivals of renewals, etc.). If all the input processes translating the system from state to state are Poisson, then the process developing in the system is Markovian [5]. This is only natural, since the Poisson process is 'memoryless', i.e. its "future" is independent of the "past".

If a considered system is in a state $S_i$ from which it can directly pass into another state $S_j$ (the transition is shown by the arc joining $S_i$ and $S_j$), we will visualize the process as the system moved from $S_i$ to $S_j$ by a Poisson arrival process. That is, as soon as the first event of these arrivals occurs, the system "jumps" from $S_i$ to $S_j$.

[5] A Poisson character of arrivals is a sufficient but not necessary condition for a process to be Markovian.

To be more illustrative, the respective graph acquires the intensities written each beside the arc of the respective transition. Let $\lambda_{ij}$ refer to the arrival rate

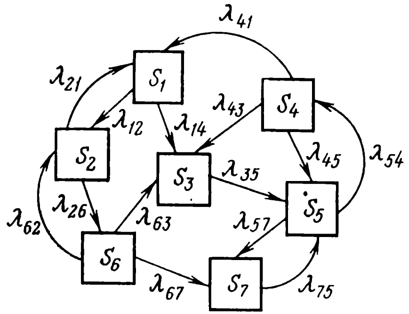


Fig. 5.3-1

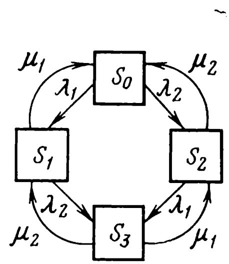


Fig. 5.3-2

moving the system from $S_i$ to $S_j$. Then a resulting graph with all arrival rates indicated is shown in Fig. 5.3-1. (We shall call a graph with all transitions indicated a directed graph, or a digraph, for short.)

Let us construct a digraph for the example of two randomly failing units in a facility, introduced in Section 5.1. As will be recalled, the states of the facility are as follows:

$S_0$ = both units are sound,
$S_1$ = first unit is being repaired, second is sound,
$S_2$ = second unit is being repaired, first is sound,
$S_3$ = both units are being repaired.

We shall compute the arrival rates translating the system from state to state under the assumption that the mean time of repair does not depend on whether only one or both units are being repaired. It will be

the case when each unit is under the care of an individual operator. Let the system be in $S_0$. Which arrival translates it in $S_1$?. It obviously is the arrival of failure occurring at the first unit. Its arrival rate $\lambda_1$ is the

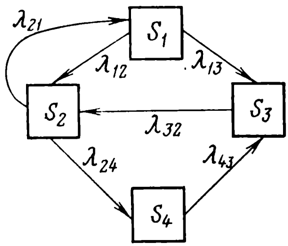


**Fig. 5.3-3**

inverse of the mean failureless time for this unit. Which process will back up the facility from $S_1$ to $S_0$? Obviously the arrivals of service completion for the first unit. The service arrival rate $\mu_1$ is the inverse of the mean time of repair for the first unit. The rates of other arrivals moving the facility through all the other arcs of Fig. 5.3-2 are computed in a similar manner.

With reference to the constructed graph we may readily develop a mathematical model of this process.

To demonstrate, consider a system having $n$ possible states $S_1, S_2, \ldots, S_n$. The probability of state $i$, $p_i(t)$, will be referred to occupying at time $t$ state $S_i$. An obvious fact is that for any time instant the sum of all state probabilities is unity, i.e.

$$\sum_{i=1}^{n} p_i(t) = 1 \tag{5.3-1}$$

With all arrival rates at hand one can obtain the probabilities of all states $p_i(t)$ as functions of time.

For this purpose the Kolmogorov equations need be derived and solved. These are special differential equations in state probabilities.

A way to deriving these equations will be demonstrated by means of an example. Let the considered system have four states $S_1$, $S_2$, $S_3$, and $S_4$; the attendant digraph is shown in Fig. 5.3-3. Consider one of the state probabilities, say $p_1(t)$. This is the probability that at time $t$ the system is in state $S_1$. Giving $t$ a small increment $\Delta t$ find the probability $p_1(t + \Delta t)$ that at the time $t + \Delta t$ the system will be in $S_1$. It may, obviously, happen in two ways: either (1) at the instant $t$ the system was in $S_1$ already and remained in this state for additional $\Delta t$, or (2) at $t$ the system was in $S_2$ and moved to $S_1$ during $\Delta t$.

Find the probability of version 1. The probability that the system at the time $t$ is in $S_1$, that is $p_1(t)$, must be multiplied by the probability that during $\Delta t$ the system will not move to either $S_2$ or $S_3$. The total input of arrivals shifting the system from $S_1$ will also be Poisson with arrival rate $\lambda_{12} + \lambda_{13}$ since a superposition of two Poisson processes yields again a Poisson process because it retains the properties of stationarity, singleness and independence. Therefore the probability that for $\Delta t$ the system will move from $S_1$ is $(\lambda_{12} + \lambda_{13})\,\Delta t$; the probability that it will not move is $1 - (\lambda_{12} + \lambda_{13})\,\Delta t$. Whence the probability of version 1 is $p_1(t)\,[1 - (\lambda_{12} + \lambda_{13})\Delta t]$.

The probability of version 2 is that at the time $t$ the system will be in $S_2$ and for $\Delta t$ will return back to $S_1$, i.e. $p_2(t)\,\lambda_{21}\Delta t$.

Summing up the probabilities of both versions (according to the rules of probability addition) yields

$$p_1(t + \Delta t) = p_1(t)\,[1-(\lambda_{12} + \lambda_{13})\Delta t] + p_2(t)\lambda_{21}\Delta t$$

Upon removing the brackets, transposing $p_1(t)$ into the left-hand side, and dividing both sides by $\Delta t$, we obtain

$$\frac{p_1(t+\Delta t)-p_1(t)}{\Delta t}=\lambda_{21}p_2(t)-(\lambda_{12}+\lambda_{13})\,p_1(t)$$

Passing to the limit by letting $\Delta t$ approach zero yields on the left-hand side the time derivative of $p_1(t)$. The resulting differential equation for $p_1(t)$ is

$$dp_1(t)/dt=\lambda_{21}p_2(t)-(\lambda_{12}+\lambda_{13})p_1(t)$$

or in a short form, without the argument $t$ of $p_1$ and $p_2$ which is no longer needed:

$$dp_1/dt=\lambda_{21}p_2-(\lambda_{12}+\lambda_{13})p_1 \qquad (5.3\text{-}2)$$

Treating the other states with similar argument, we may derive three more differential equations which when added up to (5.3-2) give the system of differential equations in state probabilities:

$$\begin{aligned}
dp_1/dt &= \lambda_{21}p_2-(\lambda_{12}+\lambda_{13})p_1\\
dp_2/dt &= \lambda_{12}p_1+\lambda_{32}p_3-(\lambda_{24}+\lambda_{21})p_2\\
dp_3/dt &= \lambda_{31}p_1+\lambda_{43}p_4-\lambda_{32}p_3\\
dp_4/dt &= \lambda_{24}p_2-\lambda_{43}p_4
\end{aligned} \qquad (5.3\text{-}3)$$

We have obtained the set of linear differential equations in four unknowns $p_1$, $p_2$, $p_3$, and $p_4$. Note that any one of them can be eliminated by invoking the normalizing condition $p_1+p_2+p_3+p_4=1$, i.e. by expressing one of the probabilities $p_i$ in terms of the others and substituting it for the respective probability in (5.3-3).

We can now summarize the general procedure of deriving these equations. The left-hand side of each of the equations must contain the derivative of the pro-

bability of state $i$; the right-hand side must contain the multiple sum of the probabilities of the states leading to the given state ($i$) times the arrival rates of respective transients minus the total intensity of all arrivals shifting the system from the considered state times the probability of this state.

Using this procedure, state the Kolmogorov equations for the system whose digraph is depicted in Fig. 5.3-2:

$$\begin{aligned} dp_0/dt &= \mu_1 p_1 + \mu_2 p_2 - (\lambda_1 + \lambda_2) p_0 \\ dp_1/dt &= \lambda_1 p_0 + \mu_2 p_3 - (\lambda_2 + \mu_1) p_1 \\ dp_2/dt &= \lambda_2 p_0 + \mu_1 p_3 - (\lambda_1 + \mu_2) p_2 \\ dp_3/dt &= \lambda_2 p_1 + \lambda_1 p_2 - (\mu_1 + \mu_2) p_3 \end{aligned} \quad (5.3\text{-}4)$$

The solution of these equations requires that some initial conditions be specified. If an initial state of the system is exactly known to be $S_i$, then at $t = 0$ $p_i(0) = 1$ and all the other initial probabilities are zeroes. So, for instance, the set (5.3-4) would be natural to solve with $p_0(0) = 1, p_1(0) = p_2(0) = p_3(0) = 0$, that is with both units sound at the initial moment of time.

Linear differential equations with constant coefficients may in general be solved analytically, but it is convenient only when the number of simultaneous equations is not higher than two or three. If more, the system is solved by numerical methods either manually or on a computer.

To sum up, the Kolmogorov equations enable us to find the probabilities of states as functions of time.

Answer now the question: what will be with the probabilities of states as $t \to \infty$? Will $p_1(t)$, $p_2(t)$, etc., tend to certain limits? If these limits exist and are independent of the initial state of the system concerned, then they are referred to as *equilibrium proba-*

*bilities*. A theorem proved in stochastic process theory states that

> if the number of states $n$ in the system is finite, and from each of them any other can be reached in a finite number of one-step transitions, then the equilibrium probabilities exist[6].

Assume that this condition is met and equilibrium probabilities exist such that

$$\lim_{t\to\infty} p_i(t) = p_i \quad (i = 1, 2, \dots n) \tag{5.3-5}$$

The equilibrium probabilities will be denoted by the same symbols $p_1$, $p_2$, etc. as the state probabilities, but one should keep in mind that these are no longer variables (time functions), rather they assume some values. Obviously, they also add to unity:

$$\sum_{i=1}^{n} p_i = 1 \tag{5.3-6}$$

Physically these probabilities are as follows. When $t$ is at infinity, the system arrives at a steady state condition in which the system may change its states at random, but their probabilities do not depend on time any longer. The equilibrium probability of state, $S_i$, may be thought of as the mean relative time the system occupies this state. To illustrate, if the system has three states $S_1$, $S_2$, and $S_3$, and their equilibrium state probabilities are 0.2, 0.3, and 0.5, respectively, this implies that in the balanced state the system on the average spends two tenths of its time in $S_1$, three tenths in $S_2$, and a half of the time in $S_3$.

A way to compute these equilibrium probabilities is rather easy. If they are constant, their derivatives must

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[6] This condition is sufficient but not necessary for equilibrium probabilities to exist.

be zero. Hence, to find equilibrium probabilities, the left-hand sides in the Kolmogorov equations should be set to zero to solve already the system of linear algebraic equations. Instead of writing the differential equations, we might have set up the system of linear equations directly with reference to the graph. Transposing the negative term in each equation from the right-hand side to the left side gives the set of equations having on the left-hand side the equilibrium probability of a given state $p_i$ multiplied by the total rate of all arrivals translating from this state; and on the right-hand side, the multiple sum of rates of all arrivals translating into the $i$th state times the probabilities of the states from which they go.

Using this rule formulate the linear equations in equilibrium probabilities for the system graphed in Fig. 5.3-2:

$$\begin{aligned}(\lambda_1+\lambda_2)p_0 &= \mu_1 p_1+\mu_2 p_2\\ (\lambda_2+\mu_1)p_1 &= \lambda_1 p_0+\mu_2 p_3\\ (\lambda_1+\mu_2)p_2 &= \lambda_2 p_0+\mu_1 p_3\\ (\mu_1+\mu_2)p_3 &= \lambda_2 p_1+\lambda_1 p_2\end{aligned} \qquad (5.3\text{-}7)$$

This system of four equations in four unknowns might seem to be easily handled. The trouble, however, is that the equations are homogeneous and hence their general solution can define the unknowns only within an arbitrary multiplier. Fortunately, we may employ the normalizing condition

$$p_0+p_1+p_2+p_3=1 \qquad (5.3\text{-}8)$$

which makes one of the equations redundant, being a linear combination of the others.

Let us solve the system (5.3-7) by specifying the following arrival and service rates: $\lambda_1 = 1$, $\lambda_2 = 2$,

$\mu_1 = 2$, $\mu_2 = 3$. Substituting the normalizing condition for the fourth line in (5.3-7) yields

$$\begin{aligned} 3p_0 &= 2p_1 + 3p_2 \\ 4p_1 &= p_0 + 3p_3 \\ 4p_2 &= 2p_0 + 2p_3 \\ 1 &= p_0 + p_1 + p_2 + p_3 \end{aligned} \qquad (5.3\text{-}9)$$

The solution to this system is

$$p_0 = 6/15 = 0.40 \quad p_1 = 3/15 = 0.20$$
$$p_2 = 4/15 \approx 0.27 \quad p_3 = 2/15 \approx 0.13$$

i.e. in the equilibrium state the system will spend on the average 40% of its time in $S_0$ (both units are healthy), 20% in $S_1$ (first unit under repair, second working), 27% in $S_2$ (second unit under repair, first working), and 13% in $S_3$ with both units under repair. The knowledge of these probabilities helps assessing the mean efficiency of the system and the load on the repair department.

Suppose that the system when in state $S_0$ (completely sound) realizes profit worth 8 units per unit time, when in $S_1$ 3 units, when in $S_2$ 5 units, and no profit when in $S_3$. The average profit per unit time in the equilibrium state will then be

$$W = 0.40 \times 8 + 0.20 \times 3 + 0.27 \times 5 = 5.15$$

We can also evaluate the utilization of repair facilities (load on operatives) handling both units 1 and 2. The fraction of time required to repair unit 1 is $p_1 + p_3 = 0.20 + 0.13 = 0.33$; the fraction of time needed to renew unit 2 is $p_2 + p_3 = 0.40$.

At this stage we may already talk of optimization of the obtained solution. Suppose we are in a capacity

to reduce mean time of repair for one unit or the other (or possibly both) for a certain cost. The question is whether or not this reduction is worth initiating. That is whether or not the increase in profit will offset the costs incurred by the repair according to new scheme.

We leave the formulation and solution of this economic problem to the reader. Major difficulties on this way are four simultaneous equations in four unknowns—but the more precious will be the final product.

# Chapter 6

# QUEUEING OR WAITING LINE THEORY

## 6.1 Objectives and Models of the Theory

Queues of customers arriving for service of one kind or another arise in many different fields of activity. Businesses of all types, government, industry, telephone exchanges, and airports, large and small—all have queueing problems. Many of these congestion situations could benefit from OR analysis which employs to this purpose a variety of queueing models, referred to as queueing systems or simply queues.

A queueing system involves a number of servers (or serving facilities) which we will call also *service channels* (in deference to the source field of the theory—telephone communication system). The serving channels can be communications links, work stations, check-out counters, retailers, elevators, buses, to mention but a few. According to the number of servers, queueing systems can be of single and multichannel type.

Customers arriving at a queueing system generally at random intervals of time are serviced generally for random times too. When a service is completed, the customer leaves the servicing facility rendering it empty and ready for a next arrival. The random nature of arrival and service times may be a cause of congestion at the input to the system at some periods when the incoming customers either queue up for service or leave the system unserved; in other periods the system

might not be completely busy because of the lack of customers, or even be idle altogether.

A queueing system operation is a random process with discrete states and continuous time. The state changes jumpwise at the instant some events occur: an arrival occurs, a service is completed, or a customer unable to wait any longer leaves the queue.

The subject matter of queueing theory is to build mathematical models which relate the specified operating conditions for the system (number of channels, their service capacity, servicing mechanism, distribution of arrivals) to the concerned characteristics of value—measures of effectiveness describing the ability of the system to handle the incoming demands. Depending on the circumstances and the objective of the study, such measures may be: the expected (mean) number of arrivals served per unit time, the expected number of busy channels, the expected number of customers in the queue and the mean waiting time for service, the probability that the number in queue is above some limit, and so on. We do not single out purposely among the given operating conditions those intended for decision variables since they may be either of these characteristics, for example, the number of channels, their capacity, service mechanism, etc. The most important part of a study is model establishment (primal problem) while its optimization (inverse problem) is intended depending on which parameters are selected to work with or to change. We are not going to consider optimization of queueing models in this text with the exception made only for the simplest queueing situations.

The mathematical analysis of a queueing system simplifies considerably when the process concerned is Markovian. As we already know a sufficient condition

for this is that all the processes changing system's states (arrival intervals, service intervals) are Poisson. If this property does not hold, the mathematical description of the process complicates substantially and acquires an explicit analytical form only in seldom cases. However, the simplest mathematics of Markov queues may prove of value for approximate handling even of those queueing problems whose arrivals are distributed not in a Poisson process. In many situations a reasonable decision on queueing system organization suffices with an approximate model.

All of the queueing systems have certain common basic characteristics. They are (a) *input process* (*arrival pattern*) which may be specified by the source of arrivals, type of arrivals and the inter-arrival times, (b) *service mechanism* which is the duration and mode of service and may be characterized by the service-time distribution, capacity of the system, and service availability, and (c) *queue discipline* which includes all other factors regarding the rules of conduct of the queue.

We start illustrating the classification breakdown with *a loss and delay system*. In a purely *loss system*, customers arriving when all the servers are busy are denied service and are lost to the system. Examples of the loss system may be met in telephony: an incoming call arrived at an instant when all the channels are busy cannot be placed and leaves the exchange unserved. In a *delay system*, an arrival incoming when all the channels are busy does not leave the system but joins the queue and waits (if there is enough waiting room) until a server is free. These latter situations more often occur in applications and are of great importance which can be readily inferred from the name of the theory.

According to the type of the *source* supplying customers to the system, the models are divided into those of a *finite population size*, when the customers are only a few, and the *infinite-population* systems. The length of the queue is a subject to further limitations imposed by allowable waiting time or handling of impatient customers which are liable to be lost to the system.

The queue discipline, that is the rule followed by the server in taking the customers in service, may be according to such self-explanatory principles as "first-come, first-served", "last-come, first-served", or "random selection for service". In some situations *priority* disciplines need be introduced to allow for realistic queues with high priority arrivals. To illustrate, in extreme cases the server may stop the service of a customer of lower priority in order to deal with a customer of high priority; this is called *preemptive* priority. For example a gantry crane working on a container ship may stop the unloading half-way and shift to another wharf to unload perishable goods of a later arrived ship. The situation when a service of a low priority customer started prior to the arrival of a high priority customer is completed and the high priority customer receives only a better position in the queue is called *non-preemptive* priority. This situation can be exemplified by an airplane which enters a queue of a few other aircraft circling around an airport and asks a permission for an emergency landing; the ground control issues the permission on the condition that it lands next to the airplane being on the runway at the moment.

Turning over to the service mechanism, we may find systems whose servicing channels are placed *in parallel* or *in series*. When in series, a customer leaving a previous server enters a queue for the next channel in

the sequence. For example, a workpiece being through the operations with one robot on a conveyor is stacked to wait when the next robot in the process is free to handle it. These operation stages of a series-channel queueing system are called *phases*.

The arrival pattern may and may not correlate with the other aspects of the system. Accordingly, the systems can be loosely divided into "open" and "close". In an open system, the distribution of arrivals does not depend on the status of the system, say on how many channels are busy. To contrast, in a close system, it does. For example, if a single operative tends a few similar machines each of which has a chance of stopping, i.e. arriving for service, at random, then the arrival rate of stoppings depends on how many machines have been already adjusted and put on or those not yet serviced. The classification examples might, of course, be added, but we constrain ourselves with those considered.

An optimization of a queueing system may be attempted from either of two standpoints, the first in favor of "queuers" or owners of the queue, and the second to favor the "queuers", i.e. the customers. The first stand makes a point of the efficiency of the system and would tend to load all the channels as high as possible, i.e. to cut down idle times. The customers on the contrary would like to cut down waiting time in a queue. Therefore, any optimization of a congestion necessitates a "systems approach" with the intrinsic complex evaluation and assessment of all consequences for each possible decision. The need for optimality over conflicting requirements may be illustrated with the viewpoint of the customer wishing to increase the number of channels which, however, would increase the total servicing cost. The development of a reason-

able model may help solving the optimization problem by choosing the number of channels with account for all pros and cons. This is the reason why we do not suggest a single measure of effectiveness for all queueing problems, formulating them instead as multiple objective problems.

All the mentioned forms of queues (and many others for which we give no room here) are being studied by queueing theory where there is a huge literature on the subject. We quote only a few books [13-17]. As a rule, the topics dealing with the theory can be found in texts on operations research [1, 6, 7]. The discussion, though, almost nowhere tunes an appropriate methodological level: the derivations are often too complicated, and the formulas are deduced not in the very best way.[1] In this (for the lack of room) brief discussion of queueing theory we are going to display two methodological approaches which substantially facilitate the process of deriving some valuable formulas. These approaches are the focus of the subsequent section.

## 6.2 The Birth and Death Process. The Little Formula

**The Birth and Death Process.** As will be recalled, with a directed graph at hand we can easily derive the Kolmogorov equations for probabilities of states to covert them to, and solve, the algebraic equations in the equilibrium state probabilities. In certain cases the latter can be solved beforehand in a general form. One of the special cases when it can be succeeded real-

[1] The author ought to confess, however, that her own book [6] unfortunately suffers from the same setback—lack of simplicity in certain derivations.

izes with the states of the considered system forming the so-called 'birth and death process'.

An illustrative graph for the birth and death process is shown in Fig. 6.2-1. This graph is outstanding in that all its vertices form a single chain and are

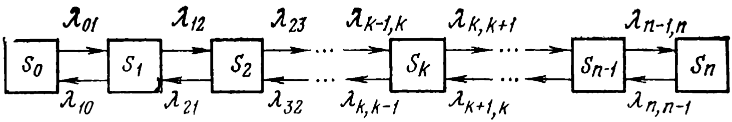


**Fig. 6.2-1**

each (except the initial and terminal vertices, $S_0$ and $S_n$) joined with each of the adjacent vertices by two (direct and reverse) arcs. The term 'birth and death process' is due to biological problems where it describes populational variations.

The birth and death model is often exploited in various applications, in particular in queueing theory, therefore it would be useful to derive the expressions for its equilibrium probabilities.

Assume that all the processes translating the system through the arcs of the graph are Poisson. Referring to the graph in Fig. 6.2-1, set up and solve the algebraic equations for the steady-state probabilities (their existence stems from the fact that each state is reachable for any other and the number of states is finite). For the first state, we have

$$\lambda_{01}p_0 = \lambda_{10}p_1 \tag{6.2-1}$$

For the second state, $S_1$:

$$(\lambda_{12} + \lambda_{10})p_1 = \lambda_{01}p_0 + \lambda_{21}p_2$$

which, in view of (6.2-1) reduces to

$$\lambda_{12}p_1 = \lambda_{21}p_2$$

and further in exactly the same manner

$$\lambda_{23}p_2 = \lambda_{32}p_3$$

so that in general

$$\lambda_{k-1,k}\ p_{k-1} = \lambda_{k,k-1}p_k$$

where $k$ assumes all integer values from 0 to $n$. Being collected together, the equilibrium probabilities $p_0$, $p_1$, . . ., $p_n$ satisfy the system

$$\begin{aligned} \lambda_{01}p_0 &= \lambda_{10}p_1 \\ \lambda_{12}p_1 &= \lambda_{21}p_2 \\ &\cdots\cdots\cdots \\ \lambda_{k-1,k}p_{k-1} &= \lambda_{k,k-1}p_k \\ &\cdots\cdots\cdots \\ \lambda_{n-1,n}p_{n-1} &= \lambda_{n,n-1}p_n \end{aligned} \tag{6.2-2}$$

which, besides, must be augmented by the normalizing condition

$$p_0 + p_1 + p_2 + \ldots + p_n = 1 \tag{6.2-3}$$

To solve the resulting set of equations, express in the first line of (6.2-2) $p_1$ via $p_0$:

$$p_1 = \frac{\lambda_{01}}{\lambda_{10}} p_0 \tag{6.2-4}$$

The second line by account of (6.2-4) yields

$$p_2 = \frac{\lambda_{12}}{\lambda_{21}}\ p_1 = \frac{\lambda_{12}\,\lambda_{01}}{\lambda_{21}\,\lambda_{10}} p_0 \tag{6.2-5}$$

and the third line, subject to (6.2-5), gives

$$p_3 = \frac{\lambda_{23}\,\lambda_{12}\,\lambda_{01}}{\lambda_{32}\,\lambda_{21}\,\lambda_{10}} p_0 \tag{6.2-6}$$

or, for any $k$ (from 1 to $n$):

$$p_k = \frac{\lambda_{k-1,\,k} \ldots \lambda_{12}\lambda_{01}}{\lambda_{k,\,k-1} \ldots \lambda_{21}\lambda_{10}}\, p_0 \tag{6.2-7}$$

We would like to attract attention to formula (6.2-7). Its numerator is the multiple of all rates attending all arcs leading from left to right, from the origin to a given state $S_k$, and the denominator contains the product of all rates attending the arcs leading from right to left (from $S_k$ to the origin).

Thus all the probabilities of states $p_0, p_1, \ldots, p_n$ are expressed in terms of one of them, viz. $p_0$. Substituting these expressions into the normalizing condition (6.2-3) and factoring out $p_0$ yields

$$p_0\left(1 + \frac{\lambda_{01}}{\lambda_{10}} + \frac{\lambda_{12}\,\lambda_{01}}{\lambda_{21}\,\lambda_{10}} + \ldots + \frac{\lambda_{n-1,\,n} \ldots \lambda_{12}\lambda_{01}}{\lambda_{n,\,n-1} \ldots \lambda_{21}\lambda_{10}}\right) = 1$$

or, solving for $p_0$:

$$p_0 = \left(1 + \frac{\lambda_{01}}{\lambda_{10}} + \frac{\lambda_{12}\,\lambda_{01}}{\lambda_{21}\,\lambda_{10}} + \ldots + \frac{\lambda_{n-1,\,n} \ldots \lambda_{12}\lambda_{01}}{\lambda_{n,\,n-1} \ldots \lambda_{21}\lambda_{10}}\right)^{-1} \tag{6.2-8}$$

Note that the coefficients of $p_0$ in each of the formulas (6.2-4)-(6.2-7) are the sequential terms in the series following the unity in (6.2-8). Hence in computing $p_0$ we have found all these coefficients.

The derived formulas are of great value in solving simplest problems of queueing theory.

**The Little Formula.** Now we denote an important formula relating (for a settled queue, i.e. that in steady state) the *expected number in system*, $L$, i.e. the expected number of customers either waiting in line and/or being serviced, and the *expected system* time, $W$, which is the time a customer spends waiting plus being serviced.

In a queueing system, which can be either of the fol-

lowing: single or multichannel, Markov or non-Markov, with unlimited or finite size of queue, we will consider two processes: that of the customers arriving at the system and that of being serviced and leaving it.

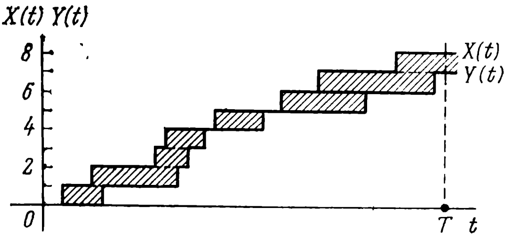


**Fig. 6.2-2**

If the queue is settled and the system is in the steady state, the mean number of arrivals per unit time (input) is equal to the number of elements leaving it (output), both the input and the output have the same rate $\lambda$.

Let $X(t)$ denote the number of customers arriving at the system up to time $t$, and $Y(t)$ the number of customers leaving it by the same instant $t$. Both functions are random and vary in jumps of unit value at the instants an arrival occurs, $X(t)$, or leaves the system, $Y(t)$. The forms of both $X(t)$ and $Y(t)$ are shown in Fig. 6.2-2; both lines vary in steps, the upper is $X(t)$, the lower is $Y(t)$. Obviously, for any moment of time $t$ their difference $Z(t) = X(t) - Y(t)$ is but the number of customers in system. When the lines $X(t)$ and $Y(t)$ merge, the system has no customers.

Consider now what happens in the long run, taking a large period of time $T$ (we can mentally extend the graph far beyond the margins of the page) and computing for it the expected number in the system. It will be given by the integral of the function $Z(t)$ over this

period divided (in order to average) by the period $T$

$$L = \frac{1}{T} \int_0^T Z(t)\,dt \qquad (6.2\text{-}9)$$

This integral can be identified as the hatched area in Fig. 6.2-2. Let us consider this figure in detail. The plot sums up of rectangles each of unit altitude and a base equal to the time the respective customer spends in system. We shall index the customers time in system as $t_1$, $t_2$, etc. Closer to the end of the period $T$ some rectangles will not completely fit the hatched area, but at a sufficiently large $T$ this will not matter. Hence we may safely equate

$$\int_0^T Z(t)\,dt = \sum_i t_i \qquad (6.2\text{-}10)$$

where the summation was performed over all the arrivals for time $T$.

Dividing both sides of (6.2-10) by the length of the period $T$ we get in view of (6.2-9):

$$L = \frac{1}{T} \sum_i t_i \qquad (6.2\text{-}11)$$

Dividing and multiplying the right-hand side of (6.2-11) by the rate $\lambda$ yields

$$L = \frac{1}{T\lambda} \sum_i t_i \lambda$$

In the quantity $T\lambda$ we can recognize the expected number of arrivals for the period $T$. If we divide the sum of all the times $t_i$ by the expected number of customers, we obtain the expected time of a customer in system, $W$. Therefore

$$L = \lambda W$$

or

$$W = L/\lambda \tag{6.2-12}$$

This result is known as *Little's formula* which states for any distribution of arrival and service times, and any service mechanism that

> the expected system time is equal to the expected number of customers in system divided by the arrival rate.

In the same manner we could derive a second Little's formula relating the *expected time in queue*, $W_q$, which is the expected time a customer spends waiting in line, and the *expected number in queue*, $L_q$, which is expected number of customers waiting to be serviced:

$$W_q = L_q/\lambda \tag{6.2-13}$$

To effect the derivation, it suffices to use as the bottom line in Fig. 6.2-2 the function $U(t)$, the number of customers that leave the queue rather than the system by time $t$ (if a customer entering the system immediately goes to service missing the queue, we may still assume the arrival joins the queue but waits in it a zero time).

The Little formulas (6.2-12) and (6.2-13) are significant results of queueing theory. Unfortunately, most texts on the subject ignore these formulas (proved in the general form relatively recently).[2]

## 6.3 Analysis of Simplest Queueing Models

This section will focus on some simplest queueing situations and their measures of effectiveness. Deducing the respective formulas we will demonstrate the main techniques characteristic of this part of the theory deal-

[2] The popular text [18] presents a different development of Little's formula as well as an easy introduction to the theory.

ing with Poisson processes. We will not aim at illustrating as many models as possible; this text indeed is not a handbook of operations research (as such we may point to [32]). Instead we intend to make the reader familiar with those "little ingenuities" which facilitate the investigation of the theory so that it does not seem a haphazard collection of examples.

In this section we will assume that all the interarrival and service times are in a Poisson process. The servicing process will be understood as a sequence of intervals resulting for the customers being served by a single continuously busy channel. About this process we may say that it has an exponential distribution or that the service-times are (negative) exponentially distributed, so we will use the names interchangeably. The required effectiveness measures for the considered queues will be introduced in the cause of discussion.

**(i) Multichannel queue having no waiting room** (Erlang loss system). This model of $n$ servers and no waiting room for any $n$ customers is one of the first, now classical problems in queueing theory. It was posed by telephone communications needs and solved at the verge of the century (1909) by A. K. Erlang for the Copenhagen Telephone Company. It is formulated as follows: $n$ channels (trunk lines) handle the demands (calls) incoming at arrival rate $\lambda$. Service (connection) is being provided at service rate $\mu$, which is the reciprocal of the mean service time. It is assumed that the calls that cannot be supplied with lines are lost. It is required to find the stationary probabilities of the states in the queue and evaluate its measures of effectiveness:

$A$ = absolute throughput (channels utilization) which is the average number of demands served per unit time in the system,

$Q$ = relative throughput which is the average proportion of arrivals served in the system,
$P_{loss}$ = probability of loss which is the probability that an arrival leaves the queue unserved,
$k$ = average number of busy channels.

SOLUTION. We will number the states of the system in accordance with the number of demands being in the system (in the circumstances it coincides with the number of busy channels) as follows:

$S_0$ = no arrivals
$S_1$ = one arrival in the system (one channel is busy, the others are idle)

. . . . . . . . . . . . . . . . . .

$S_k$ = $k$ demands in the system ($k$ channels are busy, the others free)

. . . . . . . . . . . . . . . . . .

$S_n$ = $n$ demands in the system ($n$ channels busy).

The graph for this system corresponds to that of the birth and death process. Fig. 6.3-1 shows the di-

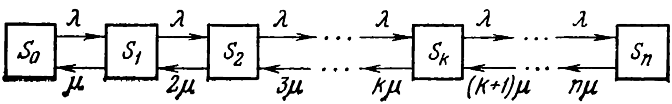


Fig. 6.3-1

graph with the corresponding rates indicated at the arcs. The transitions from $S_0$ to $S_1$ occur at the rate $\lambda$ (as soon as a demand arrives the system moves from $S_0$ to $S_1$). Generally, the system's transitions from each lower (left on the digraph) state to the next (right) state proceed at the arrival rate $\lambda$ (the upper arcs of the digraph).

Consider now the rates at the lower arcs in the digraph. Let the system be in $S_1$, i.e. with one channel busy

servicing. It is getting service to $\mu$ demands per unit time. Hence the service rate $\mu$ at the $S_1S_0$ arc. Now imagine that the system is in $S_2$ with two channels busy. To move in $S_1$ it needs that either the first or the second channel completes its service; the total rate of servicing rendered by these channels is $2\mu$; it is placed at the $S_2S_1$ arc. The total service rate of three channels is $3\mu$, and of $k$ channels $k\mu$. These rates are indicated below the bottom arcs in the digraph.

With all these rates at hand we can now make use of the expressions (6.2-7) and (6.2-8) for equilibrium probabilities in the birth and death process. From (6.2-8) we have

$$p_0 = \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{2\cdot 3\mu^3} + \dots + \frac{\lambda^k}{k!\,\mu^k} + \dots + \frac{\lambda^n}{n!\,\mu^n}\right)^{-1} \tag{6.3-1}$$

The terms of the series in the parentheses, $\lambda/\mu$, $\lambda^2/2\mu^2$, ... $\lambda^n/n!\,\mu^n$, are the coefficients of $p_0$ in the equilibrium state formulas for $p_1, p_2, \dots, p_n$:

$$\begin{gathered} p_1 = \frac{\lambda}{\mu} p_0, \quad p_2 = \frac{\lambda^2}{2\mu^2} p_0, \dots \\ p_k = \frac{\lambda^k}{k!\,\mu^k} p_0, \dots, p_n = \frac{\lambda^n}{n!\,\mu^n} p_0 \end{gathered} \tag{6.3-2}$$

Notice that the rates $\lambda$ and $\mu$ enter the expressions (6.3-1) and (6.3-2) only as the ratio $\lambda/\mu$ which is commonly denoted as

$$\rho = \lambda/\mu \tag{6.3-3}$$

and called *traffic intensity* being the ratio of effective arrival and service rates. Essentially it is the mean number of customers arriving for the time of an arrival

being served. Rewrite the last formulas in terms of this symbol:

$$p_0 = \left(1 + \rho + \frac{\rho^2}{2!} + \ldots + \frac{\rho^k}{k!} + \ldots + \frac{\rho^n}{n!}\right)^{-1} \qquad (6.3\text{-}4)$$

$$p_1 = \rho p_0,\ p_2 = \frac{\rho^2}{2!} p_0,\ \ldots,$$

$$p_k = \frac{\rho^k}{k!} p_0,\ \ldots,\ p_n = \frac{\rho^n}{n!} p_0 \qquad (6.3\text{-}5)$$

These expressions give the equilibrium probabilities of states in the system. We will employ them to deduce the performance measures of the system. First, we evaluate the probability $P_{\text{loss}}$ that an arrival will not find a free channel and be lost to the system. This condition is present when all $n$ channels are busy, whence

$$P_{\text{loss}} = p_n = \frac{\rho^n}{n!} p_0 \qquad (6.3\text{-}6)$$

This formula is often called *Erlang's loss formula.* It gives the proportion of time for which the system is fully occupied and in the telephone application, with randomly arriving calls, this is the proportion of lost calls. We may thus find the probability that a demand will be served, i.e. the relative throughput:

$$Q = 1 - P_{\text{loss}} = 1 - \frac{\rho^n}{n!} p_0 \qquad (6.3\text{-}7)$$

Multiplying $Q$ by the arrival rate $\lambda$ yields the absolute throughput

$$A = \lambda Q = \lambda\left(1 - \frac{\rho^n}{n!} p_0\right) \qquad (6.3\text{-}8)$$

Now we need only to find the mean number of busy channels $\bar{k}$. We can do it directly as the mathematical

expectation of a discrete random variable assuming the integer values in the range from 0 to $n$ with the respective probabilities $p_0, p_1, \ldots, p_n$, namely

$$\bar{k} = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + \ldots + n \cdot p_n$$

Having substituted the respective expressions for $p_k$ ($k = 0, 1, \ldots, n$) from (6.3-5), after some manipulation we could finally arrive at a correct formula for $\bar{k}$. But we shall demonstrate a more simple way to it by means of a promised "little ingenuity". Indeed, we know the absolute throughput $A$. This quantity, however, is nothing but the rate of served customers leaving the system. Each busy channel serves on the average $\mu$ customers per unit time. Hence the mean number of busy channels is

$$\bar{k} = A/\mu \qquad (6.3\text{-}9)$$

or substituting for $A$ from (6.3-8) and recalling that $\lambda/\mu = \rho$

$$\bar{k} = \rho\left(1 - \frac{\rho^n}{n!} p_0\right) \qquad (6.3\text{-}10)$$

We recommend the reader to solve a self-check example. An exchange possesses three trunk lines ($n = 3$), the arrival rate of calls put for connection is $\lambda = 1.5$ (calls per minute); the mean time to serve a call is 2 (minutes). Assuming a Poisson distribution of arrival and service times, find the steady-state probabilities and the performance measures, $A$, $Q$, $P_{\text{loss}}$, and $\bar{k}$. For the sake of comparison, the answer must be this: $p_0 = 1/13$, $p_1 = 3/13$, $p_2 = 9/26$, $p_3 = 9/26 \approx 0.346$, $A \approx 0.981$, $Q \approx 0.654$, $P_{\text{loss}} \approx 0.346$, and $\bar{k} \approx 1.96$.

These results indicate, by the way, that our exchange is substantially overloaded: of three channels about

two are busy in the mean, and about 35% of all incoming calls leave unserved. We suggest the reader evaluates how many channels (trunk lines) the exchange should possess to supply not less than 80% of subscribers with a required line. What proportion of channels will then be idle?

Here there are already some trends to optimization. To illustrate, each channel has a cost per unit time. On the other hand, each served call realizes a certain profit. Multiplying this profit by the average number of calls served per unit time, $A$, gives the mean profit rate from the system. Naturally this profit would increase with any new trunk line, but the maintenance costs of the lines also increase. Which will outbalance, the rise of profit or expenses? It depends on the conditions of the operation, i.e. the monetary rate per served call and the maintenance cost of a channel. With these values specified, one can find an optimal number of lines which would be most efficient from the economic viewpoint. We leave these computations to the reader, who is to invent and solve his own example. The invention of problems generally gives more intellectually than the solution of those posed by somebody.

**(ii) Single channel queue of unlimited size.** Single-server queues are most often met in applicational areas. The examples are a queue to a doctor, a line of football fans to a ticket seller, an on-line computer performing the subscriber demands in the time sharing mode, etc. In queueing theory these single-server models are also of special importance (most analytical expressions derived so far relate to these systems). Therefore we shall attach an especial attention to this type of queue.

Let there be a single-server queue with no constraints imposed either on the queue length or waiting time. The customers arriving at a rate $\lambda$ are serviced at a

rate $\mu$. We will seek for the equilibrium probabilities of the queue size and the performance measures:

$L$ = expected number of customers in the system,
$W$ = expected system time,
$L_q$ = expected number of customers in the queue,
$W_q$ = expected time in queue,
$P$ = probability that the server is busy (server utilization factor).

We have no need to calculate the throughputs $A$ and $Q$ since the queue has ample waiting room and each

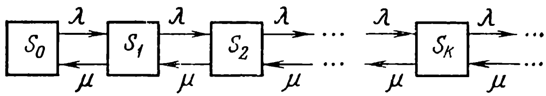


**Fig. 6.3-2**

customer sooner or later will be served, therefore $A = \lambda$, and $Q = 1$.

SOLUTION. We shall index the states of the system as before according to the number of customers in the system, viz.

$S_0$ = the server is free,
$S_1$ = the server is busy (serves an arrival), no queue,
$S_2$ = the server is busy, one customer in the queue,
. . . . . . . . . . . . . . . . . . .
$S_k$ = the server is busy, $k - 1$ customers wait in line,
. . . . . . . . . . . . . . . . . . .

Theoretically the number of states in this system is infinite. The digraph of the system is displayed in Fig. 6.3-2. It corresponds to the birth and death process with a countably infinite number of states. The arrival rate $\lambda$ moves the system from the left vertex

to the subsequent vertex at the right, and the service rate $\mu$ is responsible for the transitions in the opposite direction.

First of all we should ask ourselves whether stationary probabilities exist for this queue. The number of states is indeed infinite and principally, as $t \to \infty$, the queue can increase without limit. It indeed can take place: the equilibrium probabilities for the system exist only when it is not overloaded. It can be shown that these probabilities exist when $\rho < 1$, otherwise, at $\rho \geqslant 1$, the queue increases without bound as $t$ tends to infinity. This fact seems intuitively strange for $\rho = 1$. The system might seem to have no impossible requirements imposed, namely one customer arrives in the mean during a single service, so the conditions are for proper operation, still it does not work. An explanation is that at $\rho = 1$ the system is able to handle the arrivals only if they are regular along with the service intervals which are, due to unity traffic intensity, equal to the intervals between the successive arrivals. In this "ideal" situation, no queue will form, the channel will be continuously busy serving the arriving customers at a regular basis. Should, however, the arrival or service rate acquire a bit of randomness, the queue length will increase infinitely. We have no such situation in practice, though, because an infinite queue is an abstraction. The cost of using mathematical expectations in place of random variables may as is seen be rather high!

Now switch back to our single-channel queue with an infinite source population. Strictly speaking, the formulas for equilibrium probabilities in the birth and death process were derived only for the case of finite number of states, however, we will loosely apply them to the infinite state situation. Compute the

steady-state probabilities of the states with formulas (6.2-8) and (6.2-7). In our case the number of addends in (6.2-8) will be infinite. Thus the expression for $p_0$ will be

$$\begin{aligned} p_0 &= [1 + \lambda/\mu + (\lambda/\mu)^2 + \ldots + (\lambda/\mu)^k + \ldots]^{-1} \\ &= (1 + \rho + \rho^2 + \ldots + \rho^k + \ldots)^{-1} \qquad (6.3\text{-}11) \end{aligned}$$

The series in the parentheses is a geometrical progression. As will be recalled, it converges at $\rho < 1$ and diverges at $\rho \geqslant 1$. This is though not a strong, still an implicit proof that the equilibrium probabilities $p_0, p_1, p_2, \ldots, p_k, \ldots$ do exist only for $\rho < 1$. Summing up the series in (6.3-11) yields

$$1+\rho+\rho^2+\ldots+\rho^k+\ldots=\frac{1}{1-\rho}$$

or for (6.3-11)

$$p_0 = 1 - \rho \qquad (6.3\text{-}12)$$

The equilibrium probabilities are then found with the expressions

$$p_1 = \rho p_0, \quad p_2 = \rho^2 p_0, \quad \ldots, p_k = \rho^k p_0, \ldots$$

which, by virtue of (6.3-12) become

$$\begin{aligned} p_1 &= \rho\ (1 - \rho), \quad p_2 = \rho^2\ (1 - \rho), \ldots, \\ p_k &= \rho^k\ (1 - \rho), \ \ldots \qquad (6.3\text{-}13) \end{aligned}$$

The probabilities $p_0, p_1, \ldots, p_k$ are seen to form a geometric series with the common ratio $\rho$. However strange it might seem, the largest of them is $p_0$, that is the probability of idle channel. In other words, whatever the load of the server, if it is capable of serving the incoming arrivals ($\rho < 1$), the most probable number of customers in the system will be zero.

Obtain the average number of customers in the sytsem, $L$. This will be a cumbersome operation. The number of customers in the system is a random variable. It assumes integer values 0, 1, 2, ..., $k$, ... with probabilities $p_0$, $p_1$, $p_2$, ..., $p_k$, ..., respectively. Its mathematical expectation is

$$L = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + \ldots + k \cdot p_k + \ldots = \sum_{k=1}^{\infty} k p_k \tag{6.3-14}$$

where the summation starts from 1 rather than 0 because the first term is zero.

Substituting in (6.3-14) the expression for $p_k$ from (6.3-13) gives

$$L = \sum_{k=1}^{\infty} k \rho^k (1 - \rho)$$

or with $\rho$ $(1 - \rho)$ factored out ahead of the sum

$$L = \rho (1 - \rho) \sum_{k=1}^{\infty} k \rho^{k-1}$$

We may again exercise our ingenuity if we identify that $k\rho^{k-1}$ is but a derivative of $\rho^k$ with respect to $\rho$, hence

$$L = \rho (1 - \rho) \sum_{k=1}^{\infty} \frac{d}{d\rho} \rho^k$$

Interchanging the places of the differential and summation operators, we get

$$L = \rho (1 - \rho) \frac{d}{d\rho} \sum_{k=1}^{\infty} \rho^k \tag{6.3-15}$$

The sum in (6.3-15) can be recognized as a converging geometric series with both the first term and the common ratio being equal to $\rho$; it totals to $\rho/(1-\rho)$ and its derivative is $(1-\rho)^{-2}$. Substituting this derivative in (6.3-15) yields

$$L = \frac{\rho}{1-\rho} \tag{6.3-16}$$

Now we may invoke Little's formula (6.2-12) and find the mean time an arrival spends in the system:

$$W = \frac{\rho}{\lambda(1-\rho)} \tag{6.3-17}$$

To find the mean number of customers in the queue, $L_q$, we will think this way: the number of customers in the queue is that in the system minus the customers being served. Hence (according to the rule of adding the expectations) the expected number of customers in the queue, $L_q$, is the expected number in the system $L$, less the expected number in service. The number of customers getting service may be either zero (server is idle) or unity (server is busy). The mathematical expectation of such a random variable equals the probability that the server is busy, $P$. Obviously, $P$ equals unity minus the probability $p_0$ that the server is free:

$$P = 1 - p_0 = \rho \tag{6.3-18}$$

Consequently, the mean number of customers being served is

$$L_s = \rho \tag{6.3-19}$$

whence

$$L_q = L - \rho = \frac{\rho}{1-\rho} - \rho$$

or, finally,

$$L_q = \frac{\rho^2}{1-\rho} \tag{6.3-20}$$

Referring to the Little formula (6.2-13) find the expected time in the queue

$$W_q = \frac{\rho^2}{\lambda(1-\rho)} \tag{6.3-21}$$

By this we have found all the sought performance measures of the system.

As a self work we suggest that the reader should solve the following example. A single-channel queueing system is a railway sorting yard which handles a Poisson arrival of trains incoming with rate $\lambda = 2$ (trains per hour). A gathering of a train (servicing) takes a random time exponentially distributed with a mean of 20 minutes. The yard has waiting room for only two trains to be positioned on two body tracks; if both of them have waiting trains, the arriving trains have to await service on the approach lines. It is required to find (for the yard operating at the equilibrium conditions):

$L$ = mean number of trains bound to the yard,
$W$ = mean time a train spends in the yard (on the body tracks, on the approach lines, and in service),
$L_q$ = mean number of trains waiting in queue for gathering (no matter on what track),
$W_q$ = mean time a train waits in the queue.

Try also to evaluate:

$L_a$ = mean number of trains awaiting at the approach lines, and

$W_a$ = mean time of waiting on the approach lines (hint: this quantity is related to the previous by Little's formula).

Last but not least, find the total diurnal penalty which the yard would have to pay off for the detention of trains on the approach lines given that a train delayed on these lines by one hour incurs a penalty rate worth $a$ monetary units.

To check up, we disclose the answer: $L = 2$ (trains), $W = 1$ (hour), $L_q = 4/3$ (trains), $W_q = 2/3$ (hour), $L_a = 16/27$ (train), $W_a = 8/27 \approx 0.297$ (hour). The total diurnal penalty results by multiplying the expected number of trains arriving at the sorting yard per day, the expected waiting time on the approach lines, and the hour penalty rate $a$; for our example it is $\approx 14.2a$.

**(iii) $n$-server queueing system with unlimited queue size.** This model is developed similar to the previous for the single-channel queue, though the procedure this time will be a bit more involved. We again will index the states in accord with the number of customers in the system:

$S_0$ = no arrival (all the channels are free),
$S_1$ = one channel is busy, the others are idle,
$S_2$ = two channels are busy, the others are idle,

. . . . . . . . . . . . . . . . . . .

$S_k$ = $k$ channels are busy, the others are idle,

. . . . . . . . . . . . . . . . . . .

$S_n$ = all $n$ channels are busy (no queue),
$S_{n+1}$ = all $n$ channels are busy, one customer waits in line,

. . . . . . . . . . . . . . . . . . .

$S_{n+r}$ = all $n$ channels are busy, $r$ customers in the queue,

. . . . . . . . . . . . . . . .

The digraph for the system is depicted in Fig. 6.3-3. We leave for the reader to think over and substantiate the rates shown next to the arcs of the graph. It is a birth and death process with an infinite number of states. We give without proof the natural condition

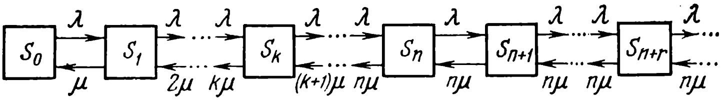


**Fig. 6.3-3**

for the equilibrium state probabilities to exist: $\rho/n < 1$. If otherwise, $\rho/n \geqslant 1$, the queue grows without bound.

Suppose the condition $\rho/n < 1$ is satisfied and the equilibrium distribution of probabilities exists. Applying the very same formulas (6.2-8) and (6.2-7) of the birth and death model, we will find these balance probabilities. The expression for $p_0$ will contain certain terms with factorials and the sum of an infinite geometric sequence with the ratio $\rho/n$:

$$p_0 = \left[1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \ldots + \frac{\rho^n}{n!} + \frac{\rho^{n+1}}{n!\,(n-\rho)}\right]^{-1}$$

$$p_1 = \frac{\rho}{1!} p_0, \ \ldots, \ p_k = \frac{\rho^k}{k!}, \ p_0, \ \ldots, \ p_n = \frac{\rho^n}{n!} p_0 \tag{6.3-22}$$

$$p_{n+1} = \frac{\rho^{n+1}}{n \cdot n!} p_0, \ \ldots, \ p_{n+r} = \frac{\rho^{n+r}}{n^r \cdot n!} p_0, \ \ldots$$

Now we will seek for the performance measures of the system. The easiest to find is the average number of busy channels $\bar{k} = \lambda/\mu = \rho$ (this is in general val-

id for any queue of unlimited length). Next in turn are the average number of customers in the system, $L$, and the average number in the queue, $L_q$. We approach firstly $L_q$, which is easier, by the expression

$$L_q = \sum_{r=1}^{\infty} r p_{n+r}$$

performing the required manipulations as in the derivation of queueing model (ii) (with differentiation of the series) to obtain

$$L_q = \frac{\rho^{n+1} p_0}{n \cdot n! \, (1-\rho/n)^2} \tag{6.3-23}$$

Adding up the average number of customers in service (which is the same as the number of busy servers) $\bar{k} = \rho$ gives

$$L = L_q + \rho \tag{6.3-24}$$

Dividing the expressions for $L$ and $L_q$ by $\lambda$, in accordance with Little's formula, leads to the mean times a customer spends in the system and in the queue

$$W = L/\lambda$$

$$W_q = L_q/\lambda \tag{6.3-25}$$

Now we solve an interesting example. A booking office with two clerks sells tickets for two destinations $A$ and $B$. This queueing system is a two-server queue with unlimited length. The passengers bound for either $A$ or $B$ arrive at an equal rate of $\lambda_A = \lambda_B = 0.45$ (passengers per minute) to build up a total arrival with the rate $\lambda_A + \lambda_B = 0.9$. It takes a clerk two minutes, on the average, to serve a passenger. From the previous operation the service is known to produce queues of complaining passengers. The man-

agement is contemplating to divide the present office into two specialized ones, selling tickets one only for $A$, the other only for $B$. We are to verify by computation whether this decision is reasonable or not. Since we know only how to evaluate the characteristics of Poisson queues, we assume that both arrivals and service processes are Poisson (this assumption will not impede the qualitative side of the analysis).

To compare, consider two modes of booking-office operation, the existing and the suggested one.

OPTION I (existing). The two-channel queueing system handles the customers arriving at the rate $\lambda = 0.9$ with the service rate $\mu = 0.5$, so that $\rho = \lambda/\mu = 1.8$. Since $\rho/2 = 0.9 < 1$, the equilibrium state distribution exists. With the first formula in (6.3-22) we find $p_0 \approx 0.0525$. The average number of customers in the queue is found by (6.3-23), viz. $L_q \approx 7.68$; the mean time in the queue (by the first formula in (6.3-25)) $W_q = 8.54$ (min).

OPTION II (suggested). Here we are to consider two single-server queues of the arrival rate $\lambda = 0.45$ and of previous $\mu = 0.5$. Since $\rho = \lambda/\mu = 0.9 < 1$, the equilibrium distribution exists. From (6.3-20), the expected queue size (of one clerk) $L_q = 8.1$.

The result is entirely unexpected. The queue grows more instead of being shorter. Perhaps the mean waiting time has been cut down? Let's have a glance. Dividing $L_q$ by $\lambda = 0.45$, we get $W_q \approx 18$ (minutes).

Just on the opposite! Instead of being relaxed, both the mean queue size and the mean waiting time in it have risen up.

Can we guess why it happened this way? Upon a little thought we might conclude that this was so because in option I (two-server queue) the average proportion of time each of the clerks was idle is shorter.

Indeed, here a clerk completing a service of a customer may begin servicing either a passenger bound for $B$ or a passenger bound for $A$. In option II, a clerk being devoid of this interchangeability, should his own queue vanish, would idly look at the passengers queueing up towards his busy servicing mate.

The reader might well agree with the explanation of the increase; but why is it that substantial? Can it not be due to a computational error? There was no error, yet the point is that both queues operate on the verge of their possibilities; should the service time increase ($\mu$ diminish) slightly, the servers would lose their control on the queue which would grow without bound. The idle time of a server is in a sense equivalent to a decrease in productivity.

So the result that seems paradoxical at first glance (or simply wrong) proves to be correct and fully explainable. Queueing theory is rich in paradoxical results of the kind whose cause is not plainly evident. The author herself would repeatedly get "surprising" results to be proved perfectly correct later.

In an attempt to drive the above situation to a positive end, the reader may argue that if a clerk sells tickets for one destination only, the service time must be cut down at least somewhat if not by half (recall that in the computation we have assumed an unchanged figure of 2 minutes for the mean service time). We suggest that such an astute reader should answer in what proportion this time must be decreased to receive a positive economic effect from the two-booking-offices decision. In this formulation we again encounter, though elementary, an optimization problem. Even with the simplest Markovian models the preparative computations help elucidating the concerned phenomenon qualitatively: which option is profitable and

which is not. The next section introduces some simple non-Markov queueing models which would widen the scope of our possibilities in the area.

Now that we hope the reader is familiar with the techniques of equilibrium probabilities evaluation and with the computations of performance measures for the simplest queueing systems (the birth and death model and Little's formula), the family of discussed queues can be augmented by two more Poisson systems suggested for self-testing.

**(iv) Single-server queueing system with limited waiting room.**

This model differs from that with unlimited queue size only in that the number of customers in the queue cannot go beyond some specified length $m$. If an arrival occurs when $m$ customers are awaiting in the queue already, it leaves the system unserved (is lost to the system). To characterize the model one needs to evaluate besides the equilibrium probabilities (by the way, in this model they exist for any $\rho$ because the number of customers in the queue is finite), the probability of being lost to the system, $P_{\text{loss}}$, the absolute throughput, $A$, the probability that the channel is busy, $P$, the mean length of the queue, $L_q$, the expected number of customers in the system, $L$, the expected waiting time in the queue, $W_q$, and the mean system time, $W$. In deriving the performance measures one may employ the same technique as for model (ii) with the difference that this time a finite rather than infinite progression must be summed up.

**(v) Machine servicing model.** Another name for this system in queueing theory literature is the *machine interference* model. It refers to the situation in which a server (or a team of servers) maintains a group of $m$

machines. Each machine is in one of two states: either "up" (running) or "down" (requiring repair service). The demands for servicing arrive at a rate $\lambda$. If the server is free, when a machine breaks down, he immediately begins the service of the machine. If a machine breaks down when the server is busy, it joins the queue and waits until the server is free. The mean time to serve a machine is $1/\mu$. The arrival rate of "demands" for attendance incoming from stopped machines to the server depends on how many machines are operating. If $k$ machines are running, then the arrival rate is $k\lambda$. To characterize the system one needs to evaluate the equilibrium probabilities of the states in the system, the mean number of running machines, and the probability that the server will be busy.

It is interesting to observe that in this queueing system the equilibrium state probabilities will exist at any values of $\lambda$ and $\mu$ because the number of states in the system is finite.

## 6.4 More Complex Queueing Models

This section will give a brief discussion on some non-Poisson queueing systems. Up to now all formulas have been derived, or at least could be derived, by the reader in command of the birth and death process and differentiation techniques. The results of this section, however, the reader will have to accept without proof.

So far we have dealt with only the simplest queueing systems for which all the processes responsible for the transitions from state to state within the system were Poisson. What will we do if they are not? How close is the assumption of the Poisson distribution models to what we have in reality? How significant are the errors

involved when its fit to the reality is not perfect? In what follows we shall attempt to answer these questions.

Unfortunately we have to openly confess that non-Markov queueing theory has no significant achievements. For non-Markov queueing systems, there exist only a few developments which enable the queue performance measures to be expressed in terms of the commonly specified conditions: number of servers, distribution of arrivals, and distribution of service times. Below we discuss some of these results.

**(i) Multichannel queueing system with no waiting room, with Poisson arrivals and an arbitrary distribution of service times.** In the previous section we deduced the Erlang formulas (6.3-4) and (6.3-5) for the equilibrium probability distribution in the loss multiserver queueing system. Of a comparatively recent origin (1959) is the work by Sevastyanov [19] who proved that these formulas are valid not only for an exponential but also for an arbitrary distribution of service times.

**(ii) Single-server queue with unlimited queue size, Poisson arrivals, and arbitrary distribution of service times.** If a single-server queueing system with an unlimited queue size has a Poisson input of rate $\lambda$, and service times are arbitrarily distributed with mean $1/\mu$ and coefficient of variation $v_\mu$, then the mean number of customers in the queue is

$$L_q = \frac{\rho^2 (1 + v_\mu^2)}{2(1-\rho)} \tag{6.4-1}$$

and the mean number of customers in the system

$$L = \frac{\rho^2 (1 + v_\mu^2)}{2(1-\rho)} + \rho \tag{6.4-2}$$

where, as before, $\rho = \lambda/\mu$, and $v_\mu$ is the ratio between the standard deviation and the expectation of service times. The above expressions are referred to as the *Pollaczek-Khinchine formulas*. Dividing $L_q$ and $L$ by $\lambda$, we obtain, by virtue of Little's formula, the mean time a customer waits in the queue and the mean time it spends in the system:

$$W_q = \frac{\rho^2 (1 + v_\mu^2)}{2\lambda (1 - \rho)} \tag{6.4-3}$$

$$W = \frac{\rho^2 (1 + v_\mu^2)}{2\lambda (1 - \rho)} + \frac{1}{\mu} \tag{6.4-4}$$

Note that in a special case when service times have an exponential distribution, $v_\mu = 1$ and the formulas (6.4-1) and (6.4-2) reduce to the familiar expressions (6.3-16) and (6.3-20) for the Poisson single-channel queue.

Take another special case when the service time is not random at all so that $v_\mu = 0$. Then the length of queue cuts twice as compared with the Poisson case. This fact has a physical explanation. If the servicing process is more ordered and runs regularly the queueing system performs better than with a poorly organized, disordered (random) service.

**(iii) Single-server service facility with arbitrary input and arbitrary service times.** The single-channel system concerned has an infinite population source supplying a recurrent input of rate $\lambda$ and coefficient of variation of inter-arrival intervals in the range between zero and unity, i.e. $0 < v_\lambda < 1$. The service times are also arbitrarily distributed with mean $1/\mu$ and coefficient of variation $v_\mu$ ranging in a zero-one interval too. Whatever explicit analytical formulas for the system have not been managed so far. We can, how-

ever, estimate the mean length of the queue by bounding it from below and from above. This estimation has been shown to work with sufficient accuracy

$$\frac{\rho^2 (v_\lambda^2+v_\mu^2)}{2(1-\rho)}+\frac{\rho(1-v_\lambda^2)}{2}$$
$$\leqslant L_q \leqslant \frac{\rho^2 (v_\lambda^2+v_\mu^2)}{2(1-\rho)}+\frac{(1-\rho)(1-v_\lambda^2)}{2} \qquad (6.4\text{-}5)$$

When the input process is Poisson, the estimates both from above and from below coincide to yield the Pollaczek-Khinchine formula (6.4-1). To effect very rough estimations, Feinberg [18] derived the simple expression

$$L_q \approx \frac{\rho^2 (v_\lambda^2+v_\mu^2)}{2(1-\rho)} \qquad (6.4\text{-}6)$$

The average number of customers in the system results simply if we add the average number of serviced customers $\rho$ to $L_q$:

$$L = L_q + \rho \qquad (6.4\text{-}7)$$

The respective mean times in queue and in system can be computed by dividing $L_q$ and $L$, in view of Little's formula, by $\lambda$.

Thus the performance measures of single-server systems with unlimited queue size may be found (if not exactly, then approximately) also in the cases where arrivals are non-Poisson and service times are not exponential.

A natural question may arise: how about the multichannel non-Poisson queues? An absolutely frank answer would be: bad. No analytic methods exist for these systems. The sole characteristic we may evaluate is the average number of busy channels $\bar{k} = \rho$. As far

as the other measures, $L_q$, $L$, $W_q$, and $W$, are concerned, they cannot enjoy any general formula.

Yet if the channels are actually many (say 5 or more), the nonexponential service times will not impede the analysis, should only the input process be Poisson. To prove, the total process of service completions rendering channels free adds up of the processes of completions in individual channels; the superposition results, as we know, in a process close to the Poisson process. Hence in this case the substitution of exponential distribution for nonexponential service times entails comparatively small errors. Fortunately, in many applications the input process is close to Poisson.

The situation is worse when arrivals are far from Poisson. This is a field for a researcher to display his ingenuity. One possibility is to assign two single-server systems of which one is "superior" and the other "inferior" to the queue at hand (the comparison may be in terms of queue length and waiting time). For single-server systems, however, we are able already to evaluate the performance measures in any circumstances.

The "inferior" and "superior" single-server systems may be devised in a variety of ways. For example, an inferior queue results if a given $n$-channel system is partitioned into $n$ single-channel queues with the total arriving input distributed between these systems in the order of arrival of the customers, that is, the first customer to the first system, the second to the second, and so on. We know that each of the systems will then receive an $n$-Erlangian input with coefficient of variation $1/\sqrt{n}$. The coefficient of variation for service times remains the same. We know also how to deduce the system time for such a single-server queue, $W$; it will be automatically longer than that for the original $n$-channel system. With this time on hand we

may give a "pessimistic" estimate also for the mean number of customers in the queue resorting to Little's formula and multiplying the mean time by the arrival rate $\lambda$ of the total input process.

An "optimistic" estimate results if the $n$-channel system concerned is modeled by one single-channel queue with service rate $n$ times quicker than in the original system, that is $n\mu$. Naturally, parameter $\rho$ must also be diminished $n$ times. The waiting time in the system decreases in this queueing system because each service is performed $n$ times quicker. Using the adjusted value $\tilde{\rho} = \rho/n$, the coefficient of variation for arrivals, $v_\lambda$, and the coefficient of variation for service times, $v_\mu$, we can approximately compute the mean number of customers in the system $\tilde{L}$. By subtracting from this value the original (nonadjusted) value for $\rho$ we obtain the mean number in the queue $\tilde{L}_q = \tilde{L} - \rho$. Both characteristics will have lower values than they would in the original $n$-channel system (as if estimated on an "optimistic" basis). Dividing them by $\lambda$ we can find the respective "optimistic" estimates for the system time and time in queue.

# Chapter 7

# STATISTICAL MODELING OF RANDOM PROCESSES (THE MONTE CARLO METHOD)

## 7.1 Idea, Purpose and Scope of the Method

In the previous chapters we have learnt to set up some analytical models of operations under stochastic uncertainty. These models enable us to establish an analytical (formal) dependence between the operation conditions, decision variables and the result (outcome) of the operation that is characterized by one or more performance measures. In many operations, queueing (or other similar) systems (e.g., devices with units that may fail) appear as "subsystems" or "components" of the controlled system. The benefits and desirability of developing the analytical models (if only approximate ones) are beyond any doubt. Unfortunately, these are amenable to construction solely for the simplest, "unpretentious" systems, and above all, they call for the assumption that the process is Markovian, which is not always the case in reality. Where analytical methods are inapplicable (or their precision is questionable) one has to rely on the universal method of statistical modeling, or as it is often called the *Monte Carlo method*.

The underlying idea of the technique is extremely simple and is as follows. Instead of describing a process using an analytical tool (differential or algebraic equations), we "play off" a random phenomenon using a specially organized procedure involving the random-

ness and yielding a random result. In actual fact, the random process is realized in each specific case in a somewhat different way; the statistical modeling, too, gives each time a new, different realization of the process examined. What may it give us? As such, next to nothing, just as, say, one case of recovery due to some medicine (or despite of it). But if these realizations are many, that's another pair of shoes. This set of realizations may be utilized as some artificially obtained statistical evidence, that lends itself to processing with conventional techniques of mathematical statistics. This processing may yield (of course, approximately) any statistic of interest: probabilities, expectations, and variances of random variables, and so forth. Monte Carlo simulation of random phenomena uses the very chance as a tool, makes it to "work for us".

Not infrequently such a procedure appears simpler than attempts to build an analytical model. With complicated, clearly non-Markovian operations involving a ghost of elements (machinery, people, organizations, auxiliary facilities) with random aspects inextricably entwined, statistical modeling appears, as a rule, far simpler than the analytical one (and often the only one possible).

In essence, the Monte Carlo technique can work out any stochastic problem, but it is only justified where the game procedure is simpler as compared with the analytical calculation. We will cite an example of a case where the Monte Carlo method is possible, but utterly pointless. Let three independent shots be fired at a target so that each shot hits with probability 1/2. It is desired to find the probability of at least one hit. Straightforward computation gives the probability to be $1 - (1/2)^3 = 7/8$. The problem, in principle,

may also be solved using a chance sampling, or statistical modeling. We will instead toss "three coins", taking heads to mean "hits", and tails "misses". A trial is considered "successful" if at least one of the coins shows head. We will perform very many trials, count "successes" and divide the result by the number $N$ of trials carried out. We will thus obtain the frequency of the event, which at a large number of trials is close to probability. What of it? Such a procedure might only be used by a person innocent to probability, though it is possible in principle[1].

And now consider another problem. Let a multichannel queueing system be given such that the process in it be clearly non-Markovian: the intervals between demands have a non-exponential distribution, the servicing times distributed similarly. What is more, channels fail now and then and must be repaired; both the failing and repair times have non-exponential distributions. We would like to find the queueing system characteristics: state probabilities as time functions, the average queue length, the average system time for a demand, and so forth. It might appear that the problem is not so involved. But anyone with a knowledge of queueing theory would not hesitate to select for its solution the method of statistical modeling (the prospects of developing here a workable analytical model are fairly bleak). He would have to simulate a multitude of realizations of the random process (to be sure, on a computer, not manually) and from such artificial "statistics" aprroximately deduce the desired probabilities (as frequencies of appropriate events) and expectations (as arithmetic means of random variables).

[1] Something of the sort happens sometimes when people with only a smattering knowledge of probability use the Monte Carlo simulation on problems having an analytical solution.

The OR problems use the Monte Carlo method in three principal roles:

(i) in the modeling of challenging, complex operations involving many interacting random factors;

(ii) in checking the applicability of simpler analytical methods, and identifying the conditions of their use;

(iii) in working out corrections to analytical relationships such as "empirical formulas" in engineering.

In Section 1.3 we have already given a tradeoff of analytical and statistical models. We can now go a step further and say that a chief disadvantage of analytical models is that they unfailingly require some assumptions to be made about, say, the process being Markovian. The adequacy of these assumptions more often than not cannot be estimated without control computations, these latter being performed using the Monte Carlo method. As a figure of speach, the Monte Carlo method in OR problems might be compared to a quality control department. On the contrary, statistical models call for no serious assumptions and approximations. In principle, "crammed" into a statistical model can be virtually anything—any distribution laws, any degree of system complexity and multiplicity of its states. But the major drawback of the statistical models is their unwieldiness and arduousness. A sea of realizations, that are necessary to find the desired parameter to an acceptable accuracy, call for far too much computer time. Moreover, the results of statistical modeling are considerably more difficult to grasp than the computations following the analytical models, and, accordingly, the solution is far more difficult to optimize (it has to be groped for blindfold). An appropriate combination of analytical and statistical methods in operations research is a matter of

skill, intuition, and experience on the part of the researcher. Sometimes analytical methods may be of help in describing some "subsystems" within a larger system for these submodels to be used as "building blocks" to erect then the building of a larger, and more complex model.

## 7.2 Organizing a Random Sampling Mechanism

The main constituent element of a statistical model is *one random realization* of the phenomenon being modeled, e.g., "one case of a machine operating to failure", "one working day of a machine shop", "one epidemics", and so on. A realization is, as it were, a "copy" of the random event with its peculiar chance nature. It is by their nature that realizations differ from one another.

An individual realization is staged using a specially developed chance mechanism (algorithm) or chance trial. Each time where the course of an event is dictated by chance, its influence is taken into account not by computation, but by a chance trial.

We elucidate the notion of the chance trial. Let a process come to a point when its further development (and hence its result) is dependent on whether or not a certain event $A$ has occurred. For example, whether a shell has hit the target, or whether apparatus is in order, or whether an object has been detected, or whether a failure has been corrected. Then, by performing a chance trial we have to answer the question of whether or not the event has occurred. How can you possibly realize the trial? You will have to utilize some mechanism of random sampling (e.g., to toss a coin, or cast a die, or pull out a badge with a number from a rotating drum, or take at random a number from a table). We are well familiar with some of the mechanisms of ran-

dom sampling (e.g., a tombola). If the trial is carried out in order to find out if the event $A$ has occurred it is to be so organized that the conditional result of the game has the same probability with the event $A$. How this is done will be seen below. Apart from random events, the course and outcome of the operation may be influenced by a variety of random variables, e.g., the time to failure of a device; the service time of a channel in queueing system; the size of a part; the weight of a train arriving to a location; the coordinates of a point hit by a shell, and so forth. Using the chance trial we can "play" both the value of any random variable and a set of values of several variables.

We will refer as random sampling to any chance experiment with a random outcome that answers one of the following questions:

1. Has the event $A$ occurred or not?
2. Which of the events $A_1, A_2, \ldots, A_k$ has occurred?
3. What value did the random variable $X$ take on?
4. What population of values did the system of random variables $X_1, X_2, \ldots, X_k$ take on?

Any realization of the random event by the Monte Carlo method involves a chain of random samplings alternating with conventional calculations. These take into account the influence of the sampling outcome on the further course of the event (in particular, on the conditions under which the next sampling will be taken).

A chance trial can be obtained in a number of ways, but there is one standard mechanism allowing any form of sampling to be realized. Namely, for each form it is sufficient to be able to obtain a random number $R$ whose values from 0 to 1 are equiprobable[2]. The va-

[2] Or rather have the same probability density.

riable $R$ will be called for short the "random number from 0 to 1". We will show that using this number we are able to stage any of the four kinds of sampling mechanism.

**1. Has the event $A$ occurred or not?** For this question to be answered requires a knowledge of the probability $p$ of the event $A$. We will play the random number $R$ from 0 to 1, and if it appears smaller than $p$, as shown

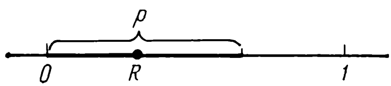


**Fig. 7.2-1**

in Fig. 7.2-1, we will consider that the event has occurred, and if more than $p$, has not.

And how do we go about it, the reader would ask, if the number $R$ is exactly $p$? The probability of such a coincidence may be disregarded. And if it has occurred, go as you please: you may either take any "equal" to mean "more" or "less", or alternatively—this will essentially have no effect on the result of the modeling.

**2. Which of the events $A_1, A_2, \ldots, A_k$ has occurred?** Let the events $A_1, A_2, \ldots, A_k$ be incompatible and form a complete group. Their probabilities $p_1, p_2, \ldots, p_k$ will then add up to unity. We break the interval (0, 1) down into $k$ sections of length $p_1, p_2, \ldots, p_k$ (Fig. 7.2-2). The number $R$ falling on a section will imply that a respective event has occurred.

**3. What value did the random variable $X$ take on?** If the random variable $X$ is discrete, i.e. it assumes the values $x_1, x_2, \ldots, x_k$ with respective probabilities $p_1$, $p_2, \ldots p_k$, then the case clearly reduces to the preceding one. We will now consider the case where the ran-

dom variable is continuous and has a predetermined probability density $f(x)$. For its value to be played it is sufficient to perform the following: to pass from the

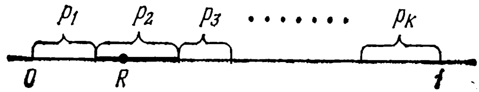


**Fig. 7.2-2**

probability density $f(x)$ to the distribution function $F(x)$ according to the relationship

$$F(x) = \int_{-\infty}^{x} f(x)\,dx, \tag{7.2-1}$$

and then to find for the function $F$ its inverse function $\Psi$. We next play the random number $R$ from 0 to 1 and find its inverse function

$$X = \Psi(R) \tag{7.2-2}$$

It is trivially shown (we will not do it) that the resultant value of $X$ has the desired distribution $f(x)$.

Figure 7.2-3 shows schematically the play (sampling) procedure for the value of $X$. Played here is the number $R$ from 0 to 1, and for it we look for a value of $X$ such that $F(X) = R$ (as shown by arrows in Fig. 7.2-3).

In actual practice we often have to play (sample) the value of a random variable having a normal distribution. For it, just as for any continuous random variable, the play rule (7.2-2) remains valid, but we may also proceed otherwise (simpler). It is common knowledge (from the Central Limit Theorem) that the addition of a sufficiently large number of independent random variables with the same distribution gives a random variable that approximately has a normal dis-

tribution. In practice, to obtain a normal distribution it suffices to add up six copies of a random number from 0 to 1. The sum of these six numbers

$$Z = R_1 + R_2 + \dots + R_6 \qquad (7.2\text{-}3)$$

has a distribution that is so close to the normal that in a majority of practical problems it may be substi-

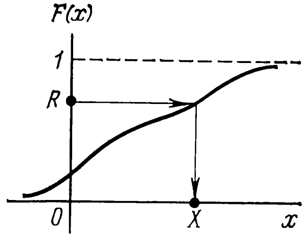


Fig. 7.2-3

tuted for the latter. For the expectation and standard deviation of this normal distribution to be equal to specified $m_X$, $\sigma_X$, we must subject the variable $Z$ to a linear transformation

$$X = \sigma_X \sqrt{2}\,(Z - 3) + m_X. \qquad (7.2\text{-}4)$$

This is exactly the desired normally distributed random variable.

4. **What manifold of values did the system of random variables $X_1, X_2, \dots, X_k$ take on?** If the random variables are independent, it is sufficient to repeat $k$ times the procedure just described. With dependent variables, each next variable must be played following its conditional distribution law, given that all the previous ones have assumed the values given by the play (we will not dwell upon the case in more detail).

We have thus considered all the four forms of sampling and seen that they all come down to the play (single or repeated) of the random number $R$ from 0 to 1.

A question arises: "But how is this number $R$ played?" There exist a number of forms of the so-called 'random number generators' used to tackle the problem. Let us dwell a bit on some of them.

The simplest of the random number generators is the rotating drum in which numbered balls (or badges) are stirred. Suppose, for example, we want to play the random number $R$ from 0 to 1 to within 0.001. We charge the drum with 1000 numbered balls, set it into rotation and, after it has stopped, select at random a ball, read its number and divide it by 1000.

We might proceed otherwise: instead of 1000 balls we put into the drum only 10, numbered 0, 1, 2, ..., 9. We draw a ball, read the first decimal place of a fraction. Return it to the drum, reset the drum into rotation, and draw a second ball—this will give us the second decimal place, and so forth. It is easily proven (we are not going to do this) that the decimal fraction thus obtained will show the uniform distribution from 0 to 1. This method has the advantage that it is in no way related to the number of decimal places to which we want to know $R$.

This invites a suggestion: not to play $R$ each time when we need it but to make it beforehand, i.e. compile a fairly extensive table listing in a random way all the figures 0, 1, 2, ..., 9 with the same probability (frequency). This idea has long been implemented: such tables have really been compiled and put into practical use. They are called random numbers tables. Sections of random numbers tables are given in many texts on probability and statistics (e.g., [20]). Also,

short passages from these tables are provided in a popular book by the present author [21] which, by the way, cites examples of the modeling of random processes using these tables.

With manual Monte Carlo simulation the random numbers tables are the best way of deriving the random number $R$ from 0 to 1. If the modeling is computer-assisted, it would be unreasonable to make use of the random numbers tables (and tables in general) as these would overburden the memory. The computer derivation of $R$ uses appropriate generators built-in into most of the computers. These may be both "physical generators" which convert random noise, and computational algorithms utilized by the computer to calculate the so-called "pseudo-random numbers". The prefix "pseudo" here stands for "as it were". Indeed, the numbers produced using these algorithms actually are not random but, for all practical purposes, these behave as such: all the values from 0 to 1 occur on average with the same frequency and, besides, there is essentially no relation between the successive values of the numbers thus obtained.

There exist a number of algorithms to derive pseudo-random numbers differing in the degree of simplicity, uniformity and other aspects (see [22]). One of the simplest algorithms for computing pseudo-random numbers consists in the following. We take two arbitrary $n$-digit binary numbers $a_1$ and $a_2$, multiply them and in the resultant product we take $n$ middle digits—this will be the number $a_3$. We next multiply $a_2$ and $a_3$ and again we take in the product $n$ middle digits, and so forth. The numbers thus derived are regarded as a sequence of binary fractions with $n$ decimal places. Such a sequence of fractions behaves as a series of values of the random number $R$ from 0 to 1. There are

other algorithms as well, which rely not on multiplication but on "summation with a shift". A more detailed discussion of concrete algorithms for deriving the pseudo-random numbers would be pointless: at present practically all the computers are either fitted with random number generators, or with tested algorithms for computing pseudo-random numbers[3].

## 7.3 Modeling a Stationary Random Process by a Single Realization

In operations research we often have to handle problems where the random process lasts rather long and under the same conditions and we are just interested in the behavior of the process in the steady state. For example, a railway sorting yard works 24 hours a day, with the intensity of the arriving traffic flow being nearly independent of time. Other examples of the systems in which a random process fairly quickly comes to a stable condition, are computers, communications lines, continuously operated apparatus, and so on.

We have already discussed the limiting stationary mode of operation and equilibrium state probabilities in Chapter 5 in connection with the Markov random processes. Do they exist for non-Markovian processes? Yes, they do exist in certain cases and they do not depend on the initial conditions. In solving the question of their existence for a given problem we can in

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[3] This argument suspiciously resembles the reasoning of a hero of *The Dunce*, a comedy by the Russian playwright Denis Fonvizin (1745-1792), who thus advocates the uselessness of geographical knowledge: " What the hell are coachmen for? It is their business. And the science is not for nobles. A nobleman will only have to say: bring me there and there, and they will bring him wherever he wishes."

a first approximation proceed as follows. We assume all the processes being Poisson. If in this case an equilibrium distribution exists, it will exist for the non-Markovian process, too. If this is the case, all the statistics for the limiting, stationary conditions may be determined by the Monte Carlo method not from a set of realizations but from one only, this realization being long enough. With this procedure a long realization provides exactly the same information about the process behavior as the set of realizations with the same total duration.

Let only one long realization of a stationary random process of total duration $T$ be available. The state probabilities of interest to us can then be found as a percentage time spent by the system in these states, and the mean values of the random variable can be obtained by the averaging not over the set of realizations but over the time along the realization.

Consider an example. A single-channel non-Markovian queue is being modeled by the Monte Carlo method. The queue has a waiting room only for two demands. A demand arrived when both places in the queue are occupied leaves the system unserved (is lost). From time to time the channel may fail. In that case, the demands in the servicing system (both being served and in the queue) do not leave the system but wait for the end of the repair. All the processes are non-Poisson but arbitrary recurrent. The possible states of the queueing system are:

$S_{0i}$ = the channel is healthy and the system contains $i$ demands,

$S_{1i}$ = the channel is being repaired, the system contains $i$ demands ($i = 0, 1, 2, 3$).

The graph of queueing system states is shown in Fig. 7.3-1. The form of the graph suggests that the

final probabilities do exist. Suppose the queueing system operation has been modeled by the Monte Carlo technique over a large time span $T$. We wish to find the characteristics of the system performance: $P_{\text{loss}}$ is

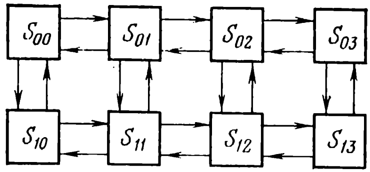


**Fig. 7.3-1**

the probability of a demand leaving the system unserved; $P_{\text{h}}$ is the probability of the channel being healthy; $A$ is the absolute throughput of the system; $L$ is the average number of demands in the system; $L_q$ is the average number of the demands in the queue; $W$ and $W_q$ are the mean times in the system and queue.

To begin with, we find the equilibrium probabilities $p_{00}, p_{01}, p_{02}, p_{03}, p_{10}, p_{11}, p_{12}, p_{13}$. To this end we calculate along the realization the total time spent by the system in each of the states, $T_{00}, T_{01}, T_{02}, T_{03}, T_{10}, T_{11}, T_{12}, T_{13}$, and divide each of them by the time $T$. We obtain

$$p_{0i} = \frac{T_{0i}}{T}, \quad p_{1i} = \frac{T_{1i}}{T} \quad (i = 0, 1, 2, 3)$$

The loss probability equals the probability of a demand arriving when there are already three demands in the system

$$P_{\text{loss}} = p_{03} + p_{13}$$

The absolute throughput is

$$A = \lambda (1 - P_{\text{loss}})$$

where $\lambda$ is the arrival rate.

The probability that the channel is healthy is derived by summing up all the probabilities whose first subscript is zero

$$P_{\text{h}} = p_{00} + p_{01} + p_{02} + p_{03}$$

The average number of demands in the system is calculated by multiplying the possible numbers in the system by the respective probabilities and summing up

$$L = 1 (p_{01} + p_{11}) + 2 (p_{02} + p_{12}) + 3 (p_{03} + p_{13})$$

This amounts to indexing on the time axis the time instants (epochs) in which the queueing system contains 0, 1, 2, 3 demands; multiplying the total length of the segments by 1, 2, 3 respectively; adding them up and dividing by $T$.

We obtain $L_q$ by subtracting from $L$ the average number of demands in service (for a single-channel queueing system this is the probability of the channel being busy)

$$P_{\text{busy}} = 1 - (p_{00} + p_{10})$$

The average time that a demand spends in the system and in the queue will be obtained by Little's formula

$$W = \frac{L}{\lambda}, \quad W_q = \frac{L_q}{\lambda}$$

This completes our brief presentation of the Monte Carlo method. The interested reader is referred to the works [6, 22], which give a more detailed treatment and consider, in particular, the accuracy of statistical modeling.

# Chapter 8

# GAME THEORY FOR DECISION MAKING

## 8.1 Subject and Problems of Game Theory

In the preceding three chapters we have considered the issues pertaining to mathematical modeling (and sometimes to solution optimization), in the cases where the operation conditions incorporate an uncertainty, relatively favorable, however, i.e. stochastic, which in principle can be refined knowing the distributions (or at least numerical characteristics) of the random factors involved.

Such an uncertainty is only half the trouble. This chapter will discuss (necessarily briefly) a far worse kind of uncertainty (in Section 2.2 we have dubbed it "unfavorable") where certain parameters contributing to the success of an operation are unknown, and there is no evidence whatsoever suggesting that these values are more probable, and those less. Uncertain (in the "unfavorable" sense) may be both external, "objective" conditions, and "subjective" ones, i.e. intentional actions of adversaries, rivals or other persons. As the saying goes "the human heart is a mistery", and the behavior of those involved may be even more difficult to predict over a random event domain.

To be sure, when dealing with an uncertain (unfavorable) situation all the inferences of a scientific study can be neither correct nor unique. But in this case, too, the quantitative analysis may be of help in arriving at a decision.

Such problems are the concern of a special branch of mathematics with a picturesque name "theory of games and decision making". In certain (rare) cases its techniques enable optimal solutions to be found. But more often than not these methods allow one just to clarify a situation, to evaluate each option from various (sometimes conflicting) angles, to analyze its advantages and disadvantages and, eventually, to make a decision, admittedly not uniquely correct, but, at least, thoroughly thought over. It is not to be forgotten that decision making under uncertainty is bound to involve some arbitrariness and risk. Shortage of information is always a problem and not an advantage (though it is in this situation that the researcher can show off his most elegant mathematical techniques). Nevertheless, in an involved situation, hard to overview as a whole, when the researcher is confronted with an abundance of detail, it is always helpful to represent the options in a form such that the selection is less arbitrary, and the risk minimal. Not infrequently the problem is formulated in such a way: What is the price to be paid for the information wanted for the operation efficiency to be improved? Note that at times the decision making does not require accurate information as to the conditions, and it suffices to indicate a "region" where they lie [23].

This chapter gives a brief treatment of the theory of games and statistical decisions. For more detailed information see [24-28].

The simplest of the situations containing the "unfavorable" uncertainty are the so-called conflict situations. The name refers to the situations with conflicting interests of two (or more) sides pursuing different (sometimes opposite) aims, with the gains of one side depending on the way in which the other will behave.

Examples of conflict situations are legion. These undoubtedly include any situation in warfare, a number of situations in competitive economies. Conflicting interests are involved in courts, sport, and species competition. To a certain degree, contradictory are interrelationships of various stages in the hierarchy of complex systems. Also, conflicting in a sense is a situation with several criteria: each places its own requirements to management and, as a rule, these requirements are conflicting.

Game theory is exactly the mathematical theory of conflicting situations. •Its objective is to work out recommendations for the reasonable behavior of the conflict sides.

Each conflicting situation taken directly from real world is extremely involved, its analysis being complicated by a number of factors of marginal importance. For a mathematical analysis of a conflict to be possible, its mathematical model is developed. This model is called a *game*.

Game differs from a real conflict in that it proceeds according to some *rules*. These rules stipulate the rights and duties of the participants, as well as the outcome of the game—who wins or loses depending on the situation at the moment. Since time immemorial humanity has been using such formalized models of conflicts—"games" in the narrow sense of the word (draughts, chess, cards, etc.). Hence the name of the theory and its terminology: the conflicting sides are called players, who "win" or "lose" in the game. We will assume that the winnings (losses) of the participants are amenable to quantitative expression (if this is not the case, then we at all times can assign it to them, e.g., in chess take a "victory" to be unity, a "loss" minus one, a "draw" zero).

In a game interests of two or more participants can clash; in the first case the game is called *two-person*, in the second an *n-person* game. Participants at an $n$-person game can form coalitions (constant or temporal). One of the tasks of game theory is the identifying of reasonable coalitions in the $n$-person game and developing of information exchange rules between the players; an $n$-person game with two constant coalitions will naturally turn into a two-person game.

The game course may be represented as a series of successive "moves" of the players. A move is a selection by a player of one of the actions stipulated by the rules and putting it into effect. Moves can be *personal* and *chance* (or *random*). With a personal move the player deliberately chooses and realizes one or another alternative (e.g., a move in chess). A chance move is made not by the player's will, but by means of some random mechanism (coin tossing, die casting, card drawing, etc.). Some games (so-called "games of pure chance") involve random moves only—these are not the concern of the theory of games. The objective of the latter is the optimization of the player's behavior in the game which (maybe in addition to chance moves) includes personal moves as well. Such games are termed strategic.

A strategy is a rule which tells a player what to do, i.e. what alternative to choose in each personal move depending on the situation at hand.

A player normally does not follow some strict, "hard and fast" rule: the selection (decision) is made in the course of the game when directly observing the situation. Theoretically, however, this makes no difference if we suppose that all of these decisions have been made beforehand by the player ("in such and such situation I will make such and such move"). This will mean that

the player has adopted a strategy. Now he may even not take part in the game personally, but to give over a list of rules to an unbiased person (umpire). The strategy can also be fed into an automat in the form of a program (it is in this way that computers play chess).

Depending on the number of strategies games are divided into *finite* and *infinite*. A game is finite if each player has at his disposal only a finite number of strategies (otherwise the game is infinite). There are games (e.g., chess) where the number of strategies is in principle finite, but so large that their complete exhaustion is practically impossible.

The *optimal* strategy is such that provides the best situation in the game, i.e. maximal payoff. If the game is repeated many times and contains, apart from the personal, the random moves as well, the optimal strategy provides the maximal average payoff.

The task of game theory is to find optimal strategies for players. The major underlying idea in finding the optimal strategies is that the opponent (in the general case, opponents) is at least as adroit as the player himself, and does his best to reach his goal. The assumption of the intelligent opponent is but one of the possible attitudes in a conflict, but the theory of games is based just on this assumption.

A game is said to be a *zero-sum* game if the total payoff to all the players is zero (i.e. each player wins at the expense of the others only). The simplest case is the *two-person zero-sum* game. The theory of these games is the most advanced section of the theory of games containing clear-cut recommendations. Below we will acquaint ourselves with some of its notions and procedures.

Game theory just as any mathematical model, is not without its constraints. One of them is the assump-

tion that the opponent (opponents) is perfectly ("ideally") intelligent. In a real conflict the optimal strategy often consists in finding out where the opponent is "silly" and making use of it to one's own advantage. Game-theoretical schemes do not include risk, that invariably accompanies reasonable decisions in real conflicts. The theory of games teaches the sides of a conflict to be most careful, overcautious. Being aware of these constraints and therefore not following blindly the recommendations derived by game theory, we can still make a reasonable use of this approach for consultations in decision making (just as a young aggressive military leader may listen to the opinion of an experienced, careful sage).

## 8.2 Matrix Games

The simplest case treated at length in game theory is the finite two-person zero-sum game (non-cooperative game of two persons or two coalitions). Consider a game $G$ with two players A and B showing conflicting interests: the gain of the one must be equal to the loss of the other. As the payoff to player A is that to playe B with the opposite sign, we may concern ourselve solely with the gain $a$ of A. It is only natural that A is out to maximize, and B to minimize $a$. For simplicity, we identify ourselves with one of the players (for definiteness, A) and will call him "we", and B, "the opponent" (it goes without saying that this gives no real odds to A). Let we have $m$ possible strategies $A_1, A_2, \ldots, A_m$, and the opponent $n$ possible strategies $B_1, B_2, \ldots, B_n$ (such a game is termed an $m \times n$ game). Denote by $a_{ij}$ our payoff in the case that we employ strategy $A_i$, and the opponent, strategy $B_j$. Suppose that we know the payoff (or average payoff)

$a_{ij}$ for each pair of strategies $A_i$, $B_j$. Then, in principle, we can set up a rectangular table (matrix) summarizing the players' strategies with respective payoffs (see Table 8.2-1),

Table 8.2-1

| | $B_1$ | $B_2$ | ... | $B_n$ |
|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | ... | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | ... | $a_{2n}$ |
| ... | ... | ... | ... | ... |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | ... | $a_{mn}$ |

If such a table is available, game $G$ is said to be reduced to matrix form (in itself, the reduction of the game to this form may be a challenging problem, that is at times practically intractable due to a spate of strategies). It is to be noted that if a game is reduced to a matrix form, then a multimove game is actually reduced to a single move: the player is only required to select a strategy. The game matrix will be denoted $[a_{ij}]$.

By way of example consider a $4 \times 5$ game in matrix form. We can select from four strategies, the opponent from five strategies. The game matrix is given in Table 8.2-2.

Let us reflect for a moment on what strategy we (player A) are to follow. Matrix 8.2-2 includes a tempting entry "10", we feel drawn to select strategy $A_3$ whereby this "titbit" will come our way. It is all very well, but the opponent is not exactly a fool! Should we choose $A_3$, he, out of spite, will choose $B_3$ that will land us a miserable "1". No, $A_3$ will not do! What then? Obviously, from the cautiousness principle (cen-

Table 8.2-2

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 3 | 4 | 5 | 2 | 3 |
| $A_2$ | 1 | 8 | 4 | 3 | 4 |
| $A_3$ | 10 | 3 | 1 | 7 | 6 |
| $A_4$ | 4 | 5 | 3 | 4 | 8 |

tral to game theory) we have to seek a strategy such that our minimal payoff be maximal. This is the so-called "minimax" principle: make the best of the worst (behavior of the opponent).

We rewrite Table 8.2-2 so that to add on the right an additional column, summarizing minimum entries of each row, and denote it by $\alpha_i$ for the $i$th row (see Table 8.2-3).

Table 8.2-3

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $\alpha_i$ |
|---|---|---|---|---|---|---|
| $A_1$ | 3 | 4 | 5 | 2 | 3 | 2 |
| $A_2$ | 1 | 8 | 4 | 3 | 4 | 1 |
| $A_3$ | 10 | 3 | 1 | 7 | 6 | 1 |
| $A_4$ | 4 | 5 | 3 | 4 | 8 | $\boxed{3}$ |
| $\beta_j$ | 10 | 8 | $\boxed{5}$ | 7 | 8 | |

Of all the values of $\alpha_i$ (the right column) we have separated the largest (3). The corresponding strategy is $A_4$. Having chosen his strategy, we can in any case be sure that (whatever the opponent's behavior) we will

win no less than 3. This is our guaranteed payoff, if we play our cards right we will not get less (and maybe even more). This payoff is called the *minorant value of the game* (or *maximin*, the maximum of minimal payoffs). We denote it by $\alpha$. In our case $\alpha = 3$.

Now, try to put ourselves into the opponent's shoes. He is not a mere pawn, and has some gumption! Choosing a strategy he would like to lose less, but must count on our worst (for him) behavior. Should he choose $B_1$, we would answer with $A_3$ and he would lose 10; if $B_2$, we would answer with $A_2$, and his loss would be 8, and so forth. Add to Table 8.2-3 one more lower row to list maxima of columns $\beta_j$. Clearly, a cautious opponent must choose a strategy minimizing this payoff (the corresponding value 5 is separated in Table 8.2-3). This $\beta$ is the payoff that a clever opponent will clearly not allow us. It is called the *majorant value of the game* (or *minimax*, the minimum of maximal payoffs). In our example $\beta = 5$ and the corresponding opponent's strategy is $B_3$.

Thus, according to the cautiousness principle (the rule: always count on the worst!), we have to select strategy $A_4$, and the opponent $B_3$. Such strategies are called *minimax* strategies (proceeding from the minimax principle). As long as both sides in our example follow their minimax strategies, the payoff will be $a_{43} = 3$.

Now suppose for a moment that we found out that the opponent follows $B_3$. Let us punish him for it and choose $A_1$ and win 5—not bad. But the opponent, too, has got his wits about him. He found that our strategy is $A_1$ and chosen $B_4$, thus reducing our payoff to 2, and so forth (the sides began to try strategies like crasy). In a word, minimax strategies in our example are unstable in relation to information about the behav-

ior of the opposite side—these strategies exhibit no equilibrium.

Is it always the case? No, not always. Look at the example with the matrix tabulated in Table 8.2-4.

Table 8.2-4

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $\alpha_i$ |
|---|---|---|---|---|---|
| $A_1$ | 2 | 4 | 7 | 5 | 2 |
| $A_2$ | 7 | $\boxed{6}$ | 8 | 7 | $\boxed{6}$ |
| $A_3$ | 5 | 3 | 4 | 1 | 1 |
| $\beta_j$ | 7 | $\boxed{6}$ | 8 | 7 | |

In this example the minorant game value is equal to the majorant one: $\alpha = \beta = 6$. What is the import of this? Minimax strategies of $A$ and $B$ will be stable. As long as both of them follow these strategies, the payoff is 6. Let us see what will occur if we (A) find out that the opponent (B) abides by $B_2$? Nothing whatsoever will change. Just because any deviation from strategy $A_2$ may only impair our situation. In exactly the same way, information obtained by the opponent will not make him abandon his strategy $B_2$. The pair of $A_2$ and $B_2$ has the property of equilibrium (balanced pair), and the payoff (in this case 6) for this pair is called the *saddle point* of matrix[1]. An indication of a saddle point

[1] The term 'saddle point' has been borrowed from geometry where it means a point of the surface at which simultaneously a minimum is achieved in one coordinate, and a maximum in the other.

and balanced pair of strategies is the equality of the minorant and majorant game values with the total value of $\alpha$ and $\beta$ being called the *value of the game.* We will denote it by $V$

$$\alpha = \beta = V \tag{8.2-1}$$

The strategies $A_i$, $B_j$ (here $A_2$, $B_2$) providing this payoff are termed the *optimal pure strategies*, and their set called the *solution* of the game. As to the game itself, it is said to be solved in this case in *pure strategies.* The both sides A and B may be supplied with optimum strategies for which their position is the best of the possible ones. If player A in the process wins 6, and player B loses 6—what of it, such are the conditions of the game: favorable for A and unfavorable for B.

The reader may ask a question: why then the optimal strategies are called "pure"? Forestalling events a bit, we answer this question: there are "mixed" strategies, too, in which the player utilizes not one strategy, but several strategies, combining them in a random way. Thus, if we assume that, apart from pure strategies, there are also mixed ones, then each finite game may have a solution, the equilibrium point. But this will be the subject of our discussion below. The presence of a saddle point in a game is not a rule, rather an exception. A majority of games have no saddle point. However, there is a class of games that always have a saddle point and, accordingly, are solved in pure strategies. These are the so-called *perfect-information* games, i.e. such games in which each player in each personal move knows all the prehistory of its development, i.e. the results of all the preceding moves, both personal and chance, Examples of perfect-information games are draughts, chess, and so forth.

It is proved in game theory that each perfect-in-

formation game has a saddle point, and hence is solved in pure strategies. In each perfect-information game there exists a pair of optimal strategies leading to a stable gain equal to the game value $V$. If such a game consists of personal moves only, then in using by each player of his optimal strategy, it is bound to end in quite a definite way—by a payoff equal to the game value. Hence, if the game solution is known the very game becomes pointless!

Take an elementary example of a perfect-information game. Two players place alternatively identical coins on a round table selecting arbitrarily the position of the coin centre (the overlapping is not allowed). The winner is the one who will place the last coin (when there will be no room for other coins). It is trivial to see that the outcome of the game is, in essence, predestined. There is a definite strategy whereby wins the player who is the first to place his coin. Namely, he must at first place his coin at the table centre and then the opponent answers to each move by a symmetrical one. It is obvious that whatever the behavior of the opponent, he cannot escape the loss. The situation with chess and perfect-information games in general is exactly the same: each of them, written in matrix form, has a saddle point, and hence a solution in pure strategies, thus it only makes sense as long as the solution is not found. For instance, a game of chess either at all times ends in a victory of White, or at all times in a victory of Black, or at all times in a draw, but which of these—we do not know as yet (fortunately for chess lovers). Moreover, it is unlikely that we will know it in the foreseeable future, as the number of strategies is ever so enormous that it is exceedingly difficult to reduce the game to a matrix form and find in it a saddle point.

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This invites the question: What is to be done if the game has no saddle point ($\alpha \neq \beta$)? Well then, if each player is committed to choose a single pure strategy, there is nothing to be done about it: one is to be guided by the minimax principle. The situation changes if we are able to mix our strategies, choosing them at random with some probabilities. The use of mixed strategies is thought in such a way: the game repeats many times; before each set, when the player can make a personal move, he relegates the choice to chance, by a kind of trial choosing the strategy that has resulted (how to organize the random mechanism we know from the preceding chapter).

In game theory mixed strategies constitute a model of variable, flexible tactics where neither of the players knows how the opponent will behave in this set. Such tactics (true, normally without any mathematical reasoning) is widely used in card games. It is to be noted here that the best way of concealing from the opponent your behavior is to make it seem random so that he cannot guess beforehand how you are going to move.

Thus, let's speak about mixed strategies. We will denote the mixed strategies of A and B respectively by $S_A = (p_1, p_2, \ldots, p_m)$, $S_B = (q_1, q_2, \ldots, q_n)$, where $p_1, p_2, \ldots p_m$ (which add up to unity) are the probabilities of player A using the strategies $A_1, A_2, \ldots, A_m$, and $q_1, q_2, \ldots, q_n$ are the probabilities of player B using the strategies $B_1, B_2 \ldots, B_n$. In the special case where all the probabilities, safe for one, are zero, and this one is unity, the mixed strategy goes over into a pure strategy.

The basic theorem of game theory states: *any finite two-person zero-sum game has at least one solution*—a pair of optimal strategies, generally mixed ($S_A^*$, $S_B^*$,) and an appropriate game value $V$.

The pair of optimal strategies $(S_A^*, S_B^*)$ constituting the game solution has the following property: if one of the players follows his optimal strategy, then the other cannot benefit by abandoning his strategy. This pair of strategies forms in the game a certain equilibrium: one player wants to maximize his payoff, the other to minimize, so that each is pulling his way with the result that, if they play their cards right, an equilibrium sets in with a stable gain $V$. If $V > 0$, then the game is favorable for us, if $V < 0$, for the opponent; at $V = 0$ the game is *fair*, i.e. equally favorable for both participants.

We now consider the example of a game without a saddle point and give (without proof) its solution. The game consists in the following: two players A and B simultaneously and arbitrarily show one, two, or three fingers. The outcome is decided by the total number of fingers shown: if it is even, A wins and receives from B a sum equal to the number; if odd, the other way round, i.e. A pays to B the sum equal to the number. What is to be recommended to the players?

Construct the game matrix. In one game each player has three strategies: to show one, two, or three fingers. The $3 \times 3$ matrix is given in Table 8.2-5, where the additional right column lists the minima of rows, and the additional lower row, the maxima of columns.

The minorant game value is $\alpha = -3$, which corresponds to strategy $A_1$. This means that with reasonable, careful behavior we may rest assured that we lose not more than 3. That's not much consolation, but still better than the payoff 5 in some matrix elements. Bad luck for us, i.e. player A... But the opponent's situation seems to be yet worse: the majorant game value is $\beta = 4$, i.e. he will lose to us a minimum

Table 8.2-5

| | $B_1$ | $B_2$ | $B_3$ | $\alpha_i$ |
|---|---|---|---|---|
| $A_1$ | 2 | −3 | 4 | $\boxed{-3}$ |
| $A_2$ | −3 | 4 | −5 | −5 |
| $A_3$ | 4 | −5 | 6 | −5 |
| $\beta_j$ | $\boxed{4}$ | $\boxed{4}$ | 6 | |

of 4 provided he behaves reasonably. The situation in general is not very bright—for either side. Let's see then if we could improve it. It appears that we could. If each side follows not some single pure strategy, but a mixed one, in which the first and third ones enter with probabilities 1/4, and the second with probability 1/2, i.e.

$$S_A^* = (1/4, 1/2, 1/4), \quad S_B^* = (1/4, 1/2, 1/4)$$

then the mean gain will be stably zero (hence the game is fair and equally favorable for either side). Strategies $S_A^*$, $S_B^*$ form the solution to the game, and its value is $V = 0$. How have we arrived at that? That is another question. We will show in the following section how finite games are solved.

## 8.3 Resolving Finite Games

Before setting out to solve an $m \times n$ game we should first of all try and simplify it by getting rid of extraneous strategies, in much the same way as we did eliminating obviously inferior alternatives in Section

2.3. We now introduce the notion of *domination.* Strategy $A_i$ of player A is said to be *dominating* over $A_k$, if row $A_i$ contains payoffs no less than appropriate entries of row $A_k$, and of these at least one is actually larger than the appropriate entry of $A_k$. If all the payoffs of $A_i$ are equal to corresponding entries in $A_k$, then strategy $A_i$ is said to be duplicating $A_k$. For strategies of player B, domination and duplication are defined similarly: *dominated* is the strategy in which all the payoffs are not larger than appropriate entries of the other strategy, and at least one of them is actually smaller; duplication implies the complete similarity between columns. Naturally, if for a strategy there is a dominating one, then the former strategy may be discarded, as may be duplicating strategies.

We clarify the above with an example. Let a $5 \times 5$ game be defined by the payoff matrix in Table 8.3-1.

To begin with, we note that strategy $A_5$ duplicates $A_2$ therefore any of them may be discarded. We notice, on discarding $A_5$, that row $A_1$ has all the entries larger than, or equal to, appropriate payoffs in $A_4$, with the result that $A_1$ will dominate over $A_4$. Discarding $A_4$ gives the $3 \times 5$ matrix of Table 8.3-2.

Table 8.3-1

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 4 | 7 | 2 | 3 | 4 |
| $A_2$ | 3 | 5 | 6 | 8 | 9 |
| $A_3$ | 4 | 4 | 2 | 2 | 8 |
| $A_4$ | 3 | 6 | 1 | 2 | 4 |
| $A_5$ | 3 | 5 | 6 | 8 | 9 |

Table 8.3-2

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|
| $A_1$ | 4 | 7 | 2 | 3 | 4 |
| $A_2$ | 3 | 5 | 6 | 8 | 9 |
| $A_3$ | 4 | 4 | 2 | 2 | 8 |

Table 8.3-3

| | $B_1$ | $B_3$ |
|---|---|---|
| $A_1$ | 4 | 2 |
| $A_2$ | 3 | 6 |
| $A_3$ | 4 | 2 |

Table 8.3-4

| | $B_1$ | $B_3$ |
|---|---|---|
| $A_1$ | 4 | 2 |
| $A_2$ | 3 | 6 |

But this is not all there is to it! A quick glance at Table 8.3-2 shows that certain strategies of B dominate over the others: for example, $B_3$ over $B_4$ and $B_5$; and $B_1$ over $B_2$ (do not forget that B seeks to lose the least!). Discarding columns $B_2$, $B_4$, and $B_5$ gives the $3 \times 2$ game of Table 8.3-3.

Lastly, in Table 8.3-3 row $A_3$ duplicates $A_1$, therefore it can be discarded. We eventually arrive at the $2 \times 2$ game of Table 8.3-4.

Try as you may, this game is not simplifiable any further. We will have to solve it. Note in passing that rejecting dominated (duplicating and clearly unfavorable) strategies in a game with a saddle point gives a solution in pure strategies. But it is a good idea to test at the beginning whether or not the game has a saddle point—this is simpler than to compare termwise all the rows and columns.

Texts on game theory generally concentrate on solving the simple $2 \times 2$, $2 \times n$, and $m \times 2$ games that permit of a geometric interpretation. But we are not going to do all that—instead we will "take the bull by the horns" and show the way in which any $m \times n$ game is solved.

Let there be a $m \times n$ game without a saddle point as given by matrix $[a_{ij}]$ of Table 8.3-5.

Table 8.3-5

| | $B_1$ | $B_2$ | ... | $B_n$ |
|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | ... | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | ... | $a_{2n}$ |
| ... | ... | ... | ... | ... |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | ... | $a_{mn}$ |

We assume that all the payoffs $a_{ij}$ are positive (this can always be achieved by adding to all the matrix elements a sufficiently large number $M$; this will increase the value of the game by $M$ leaving the solution $S_A^*$, $S_B^*$ unchanged). If all $a_{ij}$ are positive, then, of course, the game value, i.e. the mean payoff with an optimal strategy, will be positive, too.

We wish to find the solution of the game, i.e. two optimal mixed strategies

$$S_A^* = (p_1, p_2, \ldots, p_m), \qquad S_B^* = (q_1, q_2, \ldots, q_n) \tag{8.3-1}$$

which give to each side the maximally possible mean payoff (minimal loss).

We first find $S_A^*$. We know that if one of the players (here A) uses its optimal strategy, then the other (B) can in no way improve his situation by abandoning

his optimal strategy. Let us force the opponent (B) to forsake his optimal strategy by using the pure strategies $B_1, B_2, \ldots, B_n$ (while we tenaciously stick to $S_A^*$). In any event, our payoff will be no less than $V$

$$\begin{aligned} a_{11}p_1 + a_{21}p_2 + \ldots + a_{m1}p_m &\geqslant V \\ a_{12}p_1 + a_{22}p_2 + \ldots + a_{m2}p_m &\geqslant V \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}p_1 + a_{2n}p_2 + \ldots + a_{mn}p_m &\geqslant V \end{aligned} \tag{8.3-2}$$

We now divide the inequalities (8.3-2) by a positive quantity $V$ and introduce the notation

$$x_1 = p_1/V, \quad x_2 = p_2/V, \ \ldots, \quad x_m = p_m/V \tag{8.3-3}$$

The conditions (8.3-2) will then take the form

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 + \ldots + a_{m1}x_m &\geqslant 1 \\ a_{12}x_1 + a_{22}x_2 + \ldots + a_{m2}x_m &\geqslant 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{mn}x_m &\geqslant 1 \end{aligned} \tag{8.3-4}$$

where $x_1, x_2, \ldots, x_m$ are nonnegative variables. By virtue of (8.3-3) and the fact that $p_1 + p_2 + \ldots + p_m = 1$, the variables $x_1, x_2, \ldots, x_m$ meet the condition

$$x_1 + x_2 + \ldots + x_m = 1/V \tag{8.3-5}$$

Yet $V$ is nothing but our guaranteed payoff. It is only natural that we seek to make it maximal, and hence the quantity $1/V$ is minimal. The solution has thus reduced to a mathematical problem: find nonnegative values of the variables $x_1, x_2, \ldots, x_m$ such

that minimize the linear function of these variables

$$L = x_1 + x_2 + \ldots + x_m \Rightarrow \min \tag{8.3-6}$$

subject to the linear constraints (8.3-4). The reader will say, "Bah, something familiar!" Exactly—this is nothing but a *problem of linear programming*. The problem of solving the $m \times n$ game has thus reduced to a linear program with $n$ constraint inequalities and $m$ variables. Knowing $x_1, x_2, \ldots, x_m$, we can, by (9.3-3), find $p_1, p_2, \ldots, p_m$, and hence the optimal strategy $S_A^*$ and the value $V$.

The optimal strategy of player B is found in exactly the same way, with the only difference that B seeks to minimize, not maximize, the payoff, and, accordingly, to maximize (not minimize) the quantity $1/V$, and the inequalities instead of $\geqslant$ will be of $\leqslant$ type. The pair of problems of linear programming seeking the optimal strategies $(S_A^*, S_B^*)$ is called the *mutually dual problems of linear programming* (it has been proven that a maximum of a linear function in one of them is equal to a minimum of a linear function in the other so that everything is in order—we still do not arrive at different values of the game value).

Thus, solving a $m \times n$ game is equivalent to solving a problem of linear programming. It is to be noted that the opposite holds: for any problem of linear programming an equivalent game-theoretical problem can be formulated (we are not going to discuss the procedure here). This interrelation of game problems with those in linear programming appears to be of help not only for the theory of games, but for linear programming as well. The point is that there exist approximate numerical methods of solving games that in certain cases (for the problem of a larger size) turn out to be

simpler than "classical" methods of linear programming.

We will next describe one of the simplest numerical methods of game solving, the so-called *iteration method*. Its underlying idea is as follows. A "thought experiment" is played, in which the sides A and B use in turn their respective strategies seeking to win as much as possible (lose as little as possible). The experiment consists of a number of *plays*. It begins with one of the players (say, A) choosing at random one of its strategies $A_i$. The opponent (B) responds with a strategy that is the most unfavorable for $B_j$, i.e. minimizes the payoff to $A_i$. Now it is again the turn of A: he answers B following the same strategy $A_k$ that maximizes the payoff for the opponent's strategy $B_j$. Further, the move of the opponent comes. He uses his strategy that is the worst not for our last strategy $A_k$, but for a mixed strategy that includes strategies $A_i$, $A_k$, used so far, with equal probabilities. Thus at each step of the iteration process each player responds to a given move of the opponent by the strategy

> that is optimal for him with respect to a mixed strategy of the other which includes all the strategies used thus far in proportion to the frequencies of their application.

Instead of computing the mean gain each time he may simply use the gain "accumulated" throughout the preceding moves and select his strategy so that this accumulated payoff be maximal (minimal). It has been proven that such a technique "converges": in increasing the number of plays the mean payoff per play will tend to the value of the game, and the frequencies of using the strategies to their respective

probabilities in optimal mixed strategies of the players.

But the best way of grasping the iteration method is by using a specific example. We will illustrate it referring to the $3 \times 3$ game in the preceding section (Table 8.2-5). In order not to deal with negative numbers we add 5 to the matrix elements in Table 8.2-5 (see Table 8.3-6), thus increasing the value of the game by 5 with the solution $S_A^*$, $S_B^*$ remaining unchanged.

Table 8.3-6

| | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|
| $A_1$ | 7 | 2 | 9 |
| $A_2$ | 2 | 9 | 0 |
| $A_3$ | 9 | 0 | 11 |

Let's begin with some arbitrarily selected strategy of player A, e.g., $A_3$. Table 8.3-7 lists the first 15 steps of the iteration process (the reader can carry on the computation on his own). The first column contains the numbers of plays (pairs for selection) $k$; the second, number $i$ of the strategy selected in this game by the player A; the other three columns, the accumulated payoff in the first $k$ plays for the strategies used by the player in the preceding plays and strategies $B_1$, $B_2$, $B_3$ of player B in a given play (derived by adding the elements from an appropriate row to the entries of preceding row). Of these accumulated gains in Table 8.3-7 the minimal one is underscored (if they are several, all are underscored). The number underscored determines the choice by B in this play—he chooses the strategy that corresponds to the number

underscored (if there are several such numbers, any one of these is taken). We thus determine the number $j$ of the optimal (in this play) strategy $B$ (placed into the following column). The subsequent three columns list the payoffs accumulated in $k$ games respectively at strategies $A_1$, $A_2$, $A_3$ of player A (obtained by adding the elements of column $B_j$ to what was in the row above). Of these values in Table 8.3-7 the maximal have an overhead bar. It is this value that determines the choice by A of the strategy in the following play (the line below). The last three columns of Table 8.3-7

Table 8.3-7

| $k$ | $i$ | $B_1$ | $B_2$ | $B_3$ | $j$ | $A_1$ | $A_2$ | $A_3$ | $\underline{V}$ | $\overline{V}$ | $V^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 9 | $\underline{0}$ | 11 | 2 | 2 | $\overline{9}$ | 0 | 0 | 9 | 4.5 |
| 2 | 2 | 11 | $\underline{9}$ | 11 | 2 | 4 | $\overline{18}$ | 0 | 4.5 | 9 | 6.75 |
| 3 | 2 | 13 | 18 | $\underline{11}$ | 3 | 13 | $\overline{18}$ | 11 | 3.67 | 6 | 4.84 |
| 4 | 2 | 15 | 27 | $\underline{11}$ | 3 | $\overline{22}$ | 18 | $\overline{22}$ | 2.75 | 5.50 | 4.13 |
| 5 | 1 | 22 | 29 | $\underline{20}$ | 3 | 31 | 18 | $\overline{33}$ | 4.00 | 6.60 | 5.30 |
| 6 | 3 | 31 | $\underline{29}$ | 31 | 2 | $\overline{33}$ | 27 | $\overline{33}$ | 4.84 | 5.50 | 5.17 |
| 7 | 1 | 38 | $\underline{31}$ | 40 | 2 | 35 | $\overline{36}$ | 33 | 4.43 | 5.14 | 4.79 |
| 8 | 2 | $\underline{40}$ | $\underline{40}$ | $\underline{40}$ | 2 | 37 | $\overline{45}$ | 33 | 5.00 | 5.61 | 5.30 |
| 9 | 2 | 42 | 49 | $\underline{40}$ | 3 | $\overline{46}$ | 45 | 44 | 4.45 | 5.11 | 4.78 |
| 10 | 1 | $\underline{49}$ | 51 | 49 | 1 | $\overline{53}$ | 47 | $\overline{53}$ | 4.90 | 5.30 | 5.10 |
| 11 | 3 | 58 | $\underline{51}$ | 60 | 2 | 55 | $\overline{56}$ | 53 | 4.64 | 5.09 | 4.87 |
| 12 | 2 | $\underline{60}$ | $\underline{60}$ | $\underline{60}$ | 2 | 57 | $\overline{65}$ | 53 | 5.00 | 5.41 | 5.20 |
| 13 | 2 | 62 | 69 | $\underline{60}$ | 3 | $\overline{66}$ | 65 | 64 | 4.61 | 5.07 | 4.84 |
| 14 | 1 | $\underline{69}$ | 71 | $\underline{69}$ | 1 | $\overline{73}$ | 67 | 73 | 4.93 | 5.21 | 5.07 |
| 15 | 3 | 78 | $\underline{71}$ | 80 | 2 | 75 | $\overline{76}$ | 73 | 4.74 | 5.06 | 4.90 |

provide: the lower estimate of the game value $V$ that is equal to the minimal accumulated gain divided by the number of plays $k$; the upper estimate of the game value $V$ that is equal to the maximal accumulated gain, divided by $k$; and the arithmetic mean of the both (it constitutes a better estimate of the value of the game than either the minorant or the majorant estimates).

The quantity $V^*$ is seen to be varying slightly about value of the game $V = 5$ (the value of the initial game was 0, but we added 5 to each matrix element). Using Table 8.3-7 we calculate the frequencies $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{p}_3$, $\tilde{q}_1$, $\tilde{q}_2$, $\tilde{q}_3$ of strategies for the gamblers. We thus obtain

$$\begin{aligned} \tilde{p}_1 &= 4/15 \approx 0.266, & \tilde{p}_2 &= 7/15 \approx 0.468, \\ \tilde{p}_3 &= 4/15 \approx 0.266 & \tilde{q}_1 &= 2/15 \approx 0.133, \\ \tilde{q}_2 &= 8/15 \approx 0.534, & \tilde{q}_3 &= 5/15 \approx 0.333 \end{aligned}$$

which is not so much different from the values of probabilities $p_1$, $p_2$, $p_3$, $q_1$, $q_2$, $q_3$ that, as indicated above, are $1/4 = 0.25$, $1/2 = 0.50$, $1/4 = 0.25$ for the first, second, and third strategies, respectively. Such relatively good approximations have been obtained at as few as 15 iterations—not bad! Unfortunately, the approximation process proceeds further not at this pace. For the iteration method the convergence, as experience shows, is very slow. There are ways of spurring the dragging process, that lie beyond the scope of the book.

The iteration method for game solving has the very important advantage that its labour-consumption grows relatively slowly with increasing game size $(m \times n)$ whereas that of the linear programming method grows with size far more rapidly.

The reader has thus got some insight into the theory of noncooperative games and methods of solving matrix games.

A few words of caution about this theory and its applicability are in order here. At the time of its conception the theory of games was looked to for the selection of decisions in conflicting situations. The hopes were justified but to a slight degree.

Above all, in actual practice strictly non-cooperative games are a not very frequent occurrence—may be only in real games (draughts, chess, cards, etc.). Save for these artificial situations where one side seeks at all costs to maximize, and the other to minimize the payoff, such conflicts hardly can occur. It would seem that warfare is a field ideally suited for game theory. After all, here we encounter the most "fierce" antagonisms and sharply conflicting interests! But the conflict situations in this field appear to be but rarely reducible to two-person zero-sum games. Noncooperative scheme is, as a rule, only applicable to small-scale operations of marginal importance. For example, side A is a group of aircraft attacking an object, side B is the antiaircraft defence of the object, with the first seeking to maximize the probability of destroying the target, and the second to minimize. Here the scheme of the two-person zero-sum game can find its application. Now consider a more complex example: two forces of units (types of tanks, aircraft, ships) are engaged. Each side wishes to hit as many hostile combat units as possible. In that case the situation loses the purity of antagonism: it is no longer reducible to a two-person zero-sum game. If the sides' goals are not directly conflicting but just not coincident, the mathematical model becomes more involved: here we are not satisfied just with a victory, which

gives rise to a so-called "bimatrix game", where each side seeks to maximize its gain and not just to minimize the opponent's gain. The theory of such games is very much more sophisticated than that of non-cooperative games and, above all, this theory yields no distinct recommendations as to the optimal behavior of the sides [26].

The second criticism refers to the notion of 'mixed strategies'. With recurrent situations, in which each side can easily (without additional expenditures) vary its behavior from case to case, optimal mixed strategies can really improve the gain in the mean. But situations may be in which it is necessary to take only one decision (e.g., to choose a scheme of a defensive system). Would it be reasonable to "relegate the choice to chance"—roughly speaking, to toss a coin and, say, for heads select one form of the design, and for tails another? There is hardly a manager who in a complicated and crucial situation will dare to make his choice in a random manner, even if following the principles of games theory.

Lastly, one more point: it is considered in game theory that each player knows all the possible strategies of the opponent and only does not know which of them will be used in a given game. In a real world situation this is generally not the case: it is just the list of the possible strategies of the opponent that is unknown, and the best solution in a conflict situation will often be just to go beyond the strategies known to the opponent, to "bewilder" him by something new, unpredictable!

It is seen from the above that game theory has many weaknesses to serve as a basis for decision making (even in a highly conflicting situation). Is this to suggest that it is not worth studying and is of abso-

lutely no use in operations research? No, not at all. Above all, the theory is valuable in the very formulation of problems that teaches, in selecting an alternative in a conflict situation, not to forget that the opponent, too, is intelligent and to take into account his possible tricks and stratagems. Admittedly, recommendations derived from a game treatment are not always definite and not always realizable—it is still useful, in selecting a decision, to be guided, among other things, by a game model. One only should not think of the conclusions drawn from this model as definitive and unquestionable[2].

## 8.4 Statistical Decision Analysis

A close relative of game theory is the theory of statistical decision-making. It differs from game theory in that an uncertain situation has no conflict tinge—no confrontation. In problems of decision theory the unknown conditions of the operation depend not on the deliberately acting "adversary" (or other sides of the conflict), but on objective reality that in decision theory is generally referred to as "nature". Pertinent situations are often called "games with nature". "Nature" is thought of as a certain unbiased entity (in the words of A. S. Pushkin "indifferent nature"), whose "behavior" is unknown, but in any case is not malevolent.

It might appear that the absence of deliberate counteraction would simplify the choice of alternative. But it turns out that it is not so: it complicates, not

[2] The author's opinion on the role and significance of game theory is by no means generally accepted. On the contrary, some authors view game treatments in OR as basic ones (see, e.g., [28]).

simplifies. Granted the life of the decision-maker in a "game with nature" is easier indeed (nobody interferes!), but it is more difficult for him to substantiate his choice. When dealing with a conscious opponent, the element of uncertainty is partly removed by the fact that we "think" for the opponent, "make" decisions for him that are the most unfavorable for us. But in a game with nature such a concept will not do: God knows how nature is going to behave. Therefore decision theory is the most "shaky" science as far as recommendations are concerned. Still it has the right to existence and is worth attention on the side of OR people.

Consider a game with nature: we (side A) have $m$ possible strategies $A_1, A_2, \ldots, A_m$; as to the state of nature, we may make $n$ assumptions $N_1, N_2, \ldots, N_n$. Let's think of them as "strategies of nature". Our gain $a_{ij}$ for each pair of strategies $A_i$, $N_j$ is given by a matrix (Table 8.4-1).

Table 8.4-1

| | $N_1$ | $N_2$ | | $N_n$ |
|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | ... | $a_{1n}$ |
| $A_2$ | $a_{22}$ | $a_{22}$ | ... | $a_{2n}$ |
| ... | ... | ... | ... | ... |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | ... | $a_{mn}$ |

It is required to choose a strategy of player A (pure or maybe mixed, if possible) that will be more favorable as compared with the others.

On the face of it, the problem is similar to a game of two players A and N with opposite interests and is solvable by the same techniques. But this is not

quite so. The absence of the opposition on the side of nature makes the situation qualitatively different[3].

Let's give the problem some thought. The simplest case of choice in a game with nature is the case where (happily!) one of the strategies of A dominates the others, as, for instance, $A_2$ in Table 8.4-2. Here the

Table 8.4-2

| | $N_1$ | $N_2$ | $N_3$ | $N_4$ |
|---|---|---|---|---|
| $A_1$ | 1 | 2 | 3 | 5 |
| $A_2$ | 7 | 4 | 4 | 5 |
| $A_3$ | 3 | 4 | 4 | 1 |
| $A_4$ | 7 | 4 | 2 | 2 |

gain with $A_2$ for any state of nature is not less than with other strategies, and for some of them even more. It follows thus that it is precisely this strategy that must be selected.

If even the matrix of a game with nature has no dominance, it is worthwhile to see if it has duplicating strategies or strategies inferior to others under all conditions (as we did earlier reducing the game matrix). But here there is one subtlety: in such a way we can *only reduce the number of strategies of player $A$, but not of $N$*—he is absolutely indifferent as to how much we gain! Suppose that we have reduced the matrix

[3] Unfortunately, instances are not rare when those with only a smattering knowledge of operations research, when meeting in practice such a situation, forget about the "indifference" of nature and begin immediately to apply techniques of non-cooperative game theory. Such recommendations occur in texts (mostly popular science).

and now it is free of duplicating and clearly unfavorable strategies.

But what is to be used as a guidance in decision-making? Naturally, the payoff matrix $[a_{ij}]$ must be taken into account. However, the picture given by matrix $[a_{ij}]$ is in a sense incomplete and does not adequately reflect advantages and disadvantages of each decision.

Clarify this idea that is not simple at all. Suppose that payoff $a_{ij}$ for our strategy $A_i$ and state of nature $N_j$ is higher than for $A_k$ and $N_l$: $a_{ij} > a_{kl}$. But why so? Maybe we were successful in selecting the strategy $A_i$? Not necessarily. Or, maybe the state of nature $N_j$ is simply more favorable for us than $N_l$. For example, the state of nature realized in "normal conditions" is for any operation more favorable than, say, "flood", "earthquake", and so forth. It would be of help to introduce indices such that would not just give the payoff for a given situation and strategy, but also show whether the given strategy in the given situation is chosen "happily" or "unhappily".

To this end, in decision-making a notion of *risk* is introduced. The *risk premium* $r_{ij}$ of player A in using strategy $A_i$ under conditions $N_j$ is the difference between the payoff that would be obtained if we knew the conditions $N_j$ and the payoff that we will obtain not knowing them and selecting $A_i$.

It is obvious that if we (player A) knew the state of nature $N_j$, we would select a strategy maximizing our payoff. This payoff is maximal in column $N_j$. We have earlier discussed it and designated as $\beta_j$. The risk premium $r_{ij}$ is derived by deducing from $\beta_j$ the actual payoff $a_{ij}$

$$r_{ij} = \beta_j - a_{ij}. \tag{8.4-1}$$

By way of example, consider a payoff matrix $[a_{ij}]$ (Table 8.4-3) and construct for it the risk premium matrix $[r_{ij}]$ (Table 8.4-4).

Table 8.4-3

| | $N_1$ | $N_2$ | $N_3$ | $N_4$ |
|---|---|---|---|---|
| $A_1$ | 1 | 4 | 5 | 9 |
| $A_2$ | 3 | 8 | 4 | 3 |
| $A_3$ | 4 | 6 | 6 | 2 |
| $\beta_j$ | 4 | 8 | 6 | 9 |

Table 8.4-4

| | $N_1$ | $N_2$ | $N_3$ | $N_4$ |
|---|---|---|---|---|
| $A_1$ | 3 | 4 | 1 | 0 |
| $A_2$ | 1 | 0 | 2 | 6 |
| $A_3$ | 0 | 2 | 0 | 7 |

The examination of Table 8.4-4 brings out certain aspects of a given "game with nature". So, the payoff matrix $[a_{ij}]$ (Table 8.4-3) has in the second row the first and last elements equal: $a_{21} = a_{24} = 3$. These payoffs, however, are far from being equivalent for the successful selection of strategies: for the state of nature $N_1$ we could gain at most 4, and our choice $A_2$ is almost ideal; but for $N_4$ we could, by selecting $A_1$, get more by as much as 6, i.e. the choice $A_2$ is very bad. Risk is the "pay for the lack of information": in Table 8.4-4 $r_{21} = 1$, $r_{24} = 6$ (the payoffs $a_{ij}$ being

equal). To be sure, we would like to minimize the risk inherent in decision-making.

We thus have two formulations of a decision-making problem: one seeks to maximize the payoff, the other to minimize the risk.

We know that the simplest uncertainty is the stochastic uncertainty when the states of nature have some probabilities $Q_1, Q_2, \ldots, Q_n$ and these probabilities are known. It is natural then (with all the pertinent reservations in Section 2.2) to choose a strategy such that a corresponding payoff taken along the row is maximal

$$a_i = \sum_{j=1}^{n} Q_j a_{ij} \Rightarrow \max \tag{8.4-2}$$

Interestingly, the same strategy that maximizes the mean payoff also minimizes the mean risk premium

$$r_i = \sum_{j=1}^{n} Q_j r_{ij} \Rightarrow \min \tag{8.4-3}$$

so that in the case of stochastic uncertainty both approaches ("from the payoff" and "from the risk") give one and the same optimal decision.

Let's "spoil" our uncertainty a bit and suppose that the probabilities $Q_1, Q_2, \ldots, Q_n$ exist in principle, but are unknown. Sometimes in this case all the states of nature are assumed to be equiprobable (the so-called *Laplace decision criterion*, or "principle of insufficient reason"), but generally this is not to be recommended. After all, it is normally more or less clear which states are more, and which less, probable. In order to find tentative values of probabilities $Q_1, Q_2, \ldots, Q_n$ use can be made, for example, of the judgemental probability assessment (see Section 2.2). Even these tenta-

tive values are still better than the complete uncertainty. Inaccurate values of probabilities of states of nature can later be corrected by means of a specially staged experiment. The experiment can be both "ideal", i.e. completely clarifying the state of nature, and nonideal, i.e. relying on circumstantial evidence to clarify the state probabilities. Each experiment, of course, involves some expenditures, and the question thus arises: Are these expenditures compensated for by improved efficiency? It turns out that it only pays to perform the "ideal" experiment when its costs are lower than the minimal mean risk (see, e.g., [6]).

We, however, will not consider the case of stochastic uncertainty, but will take the case of "unfavorable" uncertainty, where the probabilities of the states of nature are either nonexistent, or are not susceptible to an evaluation, however approximate. The situation is thus unfavorable for a "good" decision—so we will try to find at least not the worst one.

Here everything depends on the approach to the situation, on the observer's position, on the risks involved in an unhappy decision. We describe below some of the possible approaches, points of view (or decision criteria).

**1. Maximin (Wald) decision criterion.** According to this criterion the game with nature is conducted as a game with an intelligent opponent (aggressive at that) seeking to hinder our way to success. A strategy is said to be optimal, if it guarantees a payoff in any case no less than the *minorant value of the game with nature*

$$\alpha = \max_i \min_j a_{ij} \tag{8.4-4}$$

This criterion, being the "position of extreme pessimism", dictates that one should always expect adverse

circumstances, knowing for sure that this is "rock bottom". Clearly, this approach is "overcautious", it is natural for those who are exceedingly afraid of losing. It is not the only one possible, but as an extreme case it is worth discussing.

**2. Minimax risk (Savage) criterion.** This criterion is fairly pessimistic, too. It advices, in selecting an optimal strategy, to be guided not by the payoff, but the risk. As an optimum, the strategy is selected for which the risk under the most unfavorable conditions is minimal

$$S = \min_{i} \max_{j} r_{ij} \tag{8.4-5}$$

The approach dictates that one should by all means avoid large risks in decision-making. As far as "pessimism" is concerned the Savage criterion is similar to that of Wald with the only distinction that the very pessimism is here treated otherwise.

**3. Pessimism-optimism (Hurwicz) criterion.** This criterion recommends decision makers to rely neither on complete pessimism ("always count on the worst!"), nor complete, carefree optimism ("something may turn up"). According to this criterion the strategy is selected from the condition

$$H = \max_{j} [\varkappa \min_{i} a_{ij} + (1 - \varkappa) \max_{i} a_{ij}] \tag{8.4-6}$$

where $\varkappa$ is the *coefficient of pessimism* selected between 0 and 1. At $\varkappa = 1$ the Hurwicz criterion goes over into the Wald criterion; at $\varkappa = 0$, into the "complete optimism" criterion that recommends one to select the strategy whose payoff in the row is maximal. At $0 < \varkappa < 1$ we have a happy mean. The coefficient $\varkappa$ is selected from subjective considerations: the higher

the risk involved and the more security we are seeking for and the less we are inclined to take risk, the closer to unity $\varkappa$ is chosen.

If desired, a criterion may be developed that is similar to $H$ and based not on the payoff, but on the risk, but we will not discuss it here.

The reader may very well argue: if the selection of a criterion is subjective, the selection of the coefficient $\varkappa$ is subjective, too, hence the decision is taken subjectively, too. Where is science here? What has mathematics to do with all that? Would it not be better simply, without any mathematical tricks, to make a decision arbitrarily?

In a sense the reader is right—decision making under uncertainty is always conventional and subjective. But still, to a certain (limited) degree mathematical techniques are of use here as well. Above all, they enable the game with nature to be reduced to matrix form, which is not always simple, especially when strategies are numerous (in our examples these were quite few). In addition, they allow one instead of just viewing a payoff matrix (or risk matrix), which, if large, may be simply a strain for the eyes, to introduce a consistent numerical analysis of the situation from different angles, to weigh recommendations coming from these angles and, finally, to decide on something. This is analogous to discussing a question from various stands, and a dispute, as we know, engenders a truth. Thus, do not await from decision theory some definitive, unquestionable recommendations. All it can help with is advice...

If the recommendations coming from various criteria coincide—all the better, hence you can confidently select the decision recommended—it is unlikely to fail. If then, as is often the case, recommendations

are at variance, use your brains. Reflect on these recommendations, and varied results to which they lead, finalize your point of view, and make the final selection. It is not to be forgotten that any decision problem involves a measure of arbitrariness, if only in building a mathematical model and selecting a performance measure. The mathematics used in operations research does not remove this arbitrariness, but only allows us to place it into a proper perspective.

Consider an elementary example of a $4 \times 3$ game with nature whose payoff matrix $[a_{ij}]$ is given in Table 8.4-5. We look closer at the matrix and try

Table 8.4-5

| | $N_1$ | $N_2$ | $N_3$ |
|---|---|---|---|
| $A_1$ | 20 | 30 | 15 |
| $A_2$ | 75 | 20 | 35 |
| $A_3$ | 25 | 80 | 25 |
| $A_4$ | 85 | 5 | 45 |

offhand, without calculations, to decide on a strategy to be utilized. Even for the small matrix this is not so easy.

We will now try to help ourselves using the criteria of Wald, Savage, and Hurwicz, and in the latter we will put $\varkappa = 0.6$ (a slightly pessimistic bent). What will they tell us?

1. The floor is given to Wald. Compute minima in rows (see Table 8.4-6) and select the strategy for which the row minimum is maximal (25). This is strategy $A_3$.

2. Now, the floor is given to Savage. We will go over from the payoff matrix (Table 8.4-6) to the risk

Table 8.4-6

| | $N_1$ | $N_2$ | $N_3$ | $\alpha_i$ |
|---|---|---|---|---|
| $A_1$ | 20 | 30 | 15 | 15 |
| $A_2$ | 75 | 20 | 35 | 20 |
| $A_3$ | 25 | 80 | 25 | $\boxed{25}$ |
| $A_4$ | 85 | 5 | 45 | 5 |

matrix (Table 8.4-7) and the right additional column has the maximal value of the risk in a row, $\gamma_i$.

Table 8.4-7

| | $N_1$ | $N_2$ | $N_3$ | $\gamma_i$ |
|---|---|---|---|---|
| $A_1$ | 65 | 50 | 30 | 65 |
| $A_2$ | 10 | 60 | 10 | $\boxed{60}$ |
| $A_3$ | 60 | 0 | 20 | $\boxed{60}$ |
| $A_4$ | 0 | 75 | 0 | 75 |

Of the right column entries the minimal one (60) corresponds to strategies $A_2$ and $A_3$, hence both of them are optimal according to Savage.

3. The floor is next given to Hurwicz (at $\varkappa = 0.6$). We again rewrite Table 8.4-5 but this time in such a way that in the right three additional columns we will have: the minimum of row $\alpha_i$, its maximum $\omega_i$ and the quantity $h_i = \varkappa\alpha_i + (1 - \varkappa)\,\omega_i$ rounded off to the next integer (Table 8.4-8).

The maximal value $h_i = 47$ corresponds to strategy $A_3$—a unanimous choice of all the three criteria.

Table 8.4-8

| | $N_1$ | $N_2$ | $N_3$ | $\alpha_i$ | $\omega_i$ | $h_i$ |
|---|---|---|---|---|---|---|
| $A_1$ | 20 | 30 | 15 | 15 | 30 | 21 |
| $A_2$ | 75 | 20 | 35 | 20 | 75 | 42 |
| $A_3$ | 25 | 80 | 25 | 25 | 80 | $\boxed{47}$ |
| $A_4$ | 85 | 5 | 45 | 5 | 85 | 37 |

We will now take the case where there is a "conflict" between criteria. The payoff matrix $[a_{ij}]$ (with preliminarily written columns of row minima $\alpha_i$, row maxima $\omega_i$, and values of $h_i$ (at $\varkappa = 0.6$) are given in Table 8.4-9).

Table 8.4-9

| | $N_1$ | $N_2$ | $N_3$ | $N_4$ | $\alpha_i$ | $\omega_i$ | $h_i$ |
|---|---|---|---|---|---|---|---|
| $A_1$ | 19 | 30 | 41 | 49 | $\boxed{19}$ | 49 | 31 |
| $A_2$ | 51 | 38 | 10 | 20 | 10 | 51 | 26 |
| $A_3$ | 73 | 18 | 81 | 11 | 11 | 81 | $\boxed{38}$ |

By the Wald criterion the optimal strategy is $A_1$, by the Hurwicz criterion (with $\varkappa = 0.6$) $A_3$. We will now see what the Savage criterion will tell us. The risk matrix with an additional column containing row maxima $\gamma_i$ is given in Table 8.4-10.

The minimum in the last column is 38, so that the Savage criterion, just as the Hurwicz criterion, votes for $A_3$.

Table 8.4-10

| | $N_1$ | $N_2$ | $N_3$ | $N_4$ | $\gamma_i$ |
|---|---|---|---|---|---|
| $A_1$ | 54 | 8 | 40 | 0 | 54 |
| $A_2$ | 22 | 0 | 71 | 29 | 71 |
| $A_3$ | 0 | 20 | 0 | 38 | $\boxed{38}$ |

We should take a closer look at this. If we are very much afraid of the small payoff 11 associated with strategy $A_3$, we will then select $A_1$ that is recommended by the overcautious Wald criterion, whereby we can at least guarantee the payoff 19, and maybe even more. If our pessimism is not so black, we should, it seems, decide on $A_3$ recommended by two of the three criteria.

The reader is sure to have noticed that here we are not speaking mathematical language, rather the language of consideration and common sense. That's that—an uncertainty is no good and in the absence of necessary information no mathematics will help you to decide adequately on an optimal decision. Such is our life, the future is full of uncertainty and not infrequently we have to take not strictly optimal, but "acceptable" decisions, which are worked out using various approaches and criteria that appear as conflicting sides.

To sum up: the three criteria (Wald, Savage, and Hurwicz) were all formulated for pure strategies, but each of them can be extended to include mixed ones as well in much the same way as we did in game theory. However, in games with nature the mixed strategies are of limited (notably theoretical) use. Whereas against the reasonable opponent the mixed strategies

at times are of value as a "trick" to mislead the opponent, in dealing with "indifferent nature" this argument doesn't hold. Moreover, mixed strategies make sense solely for repeated games. But when we repeat it, then stochastic behavior of the situation is bound to stand out with the result that we can make use of it to handle the problem stochastically, which, as we know, yields no mixed strategies. Furthermore, in a situation with "unfavorable" uncertainty, where we need information so badly, the main task is to glean this information and not to engineer some tricks that would enable us to do without it. One of the major applications of decision theory is precisely to *experimental design* whose objective is to clarify and refine some data. We will not discuss the issues of experimental design here: it is a separate subject worthy of serious consideration. The interested reader is referred to special texts [29, 30], and also to a readable popular book [27]. The major principle of experimental design is that any prior decision must be revised with due account for fresh information obtained.

* * *

Thus, our short account of problems, principles and methodology of operations research came to an end. The author strove to introduce the reader not only to the potentialities, but also to constraints of mathematical techniques employed in decision making. But remember—neither of these techniques saves the worker the trouble of *thinking*. Not just speculate, but to invoke mathematical calculations, remembering that, as Hamming aptly remarked, "the major objective of calculation is not figures, but understanding."

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**OPERATIONS RESEARCH**
**by E. S. Wentzel**

The aim of this book is to present in a widely appealing form the subject matter and methods of operations research (OR), a managerial tool designed to increase the effectiveness of managerial decisions as an objective supplement to the subjective assessment.
With no confinement to an exclusive area of practice, the discussion in the book is focused on the methodological aspects which are common to all OR problems wherever they might appear. Therefore the main emphasis is placed on such methodological treatises as problem formulation, model development, and assessment of computational results, rather than on mathematical rigor.

In writing this text the author—a well known Soviet mathematician and popular novelist—employed her many-year experience in governmental OR projects and lectureship at the Moscow Institute of Transportation Engineering to give a vivid presentation of the relevant methods of mathematical programming, game theory, queueing models, and decision analysis. The mathematics of the book is simple and requires only expertise in probability theory. The book is devoted primarily to the practitioners who are novices to the subject. It will be interesting to students, engineers, managerial staff and all those involved in decision making.